Finding Nessie:
The structure and origin of confined Holmboe waves

Adrien Lefauve

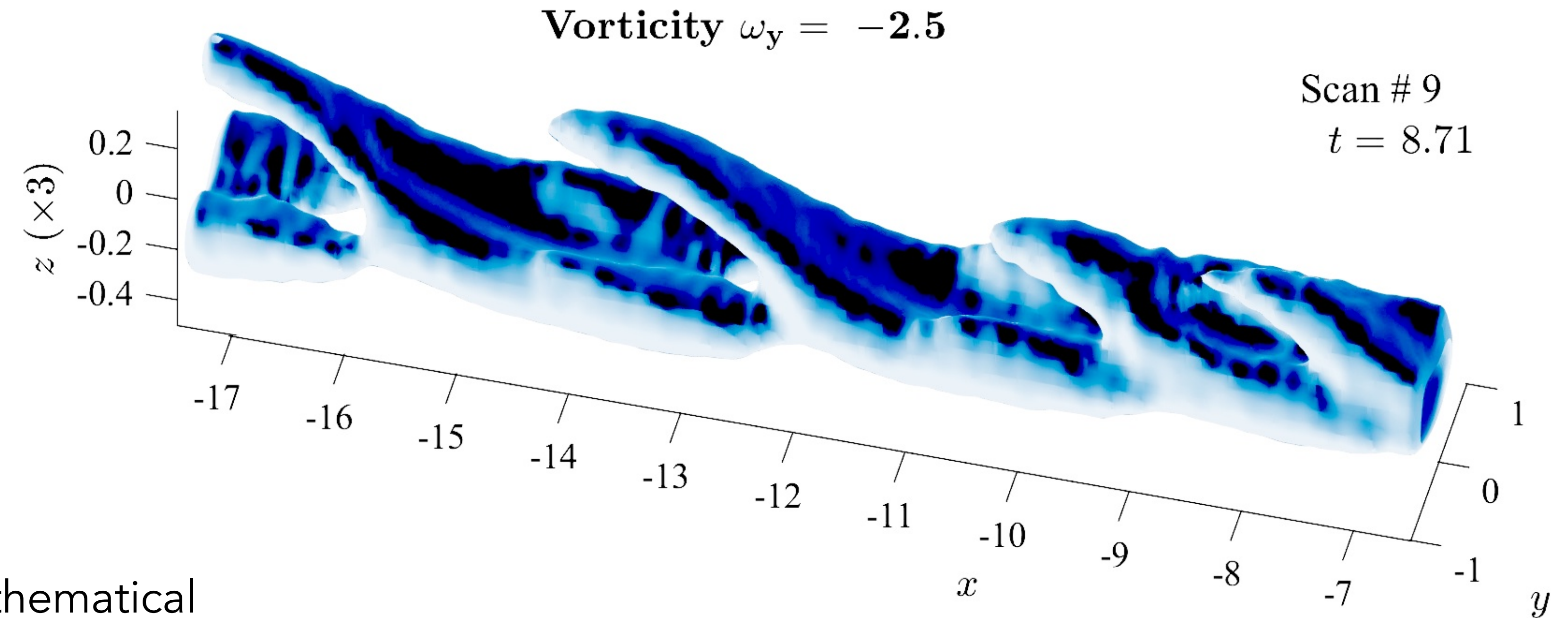
Jamie Partridge

Qi Zhou

Colm Caulfield

Stuart Dalziel

Paul Linden



MUST – Mathematical Underpinnings of Stratified Turbulence

EPSRC – Engineering and Physical Sciences Research Council

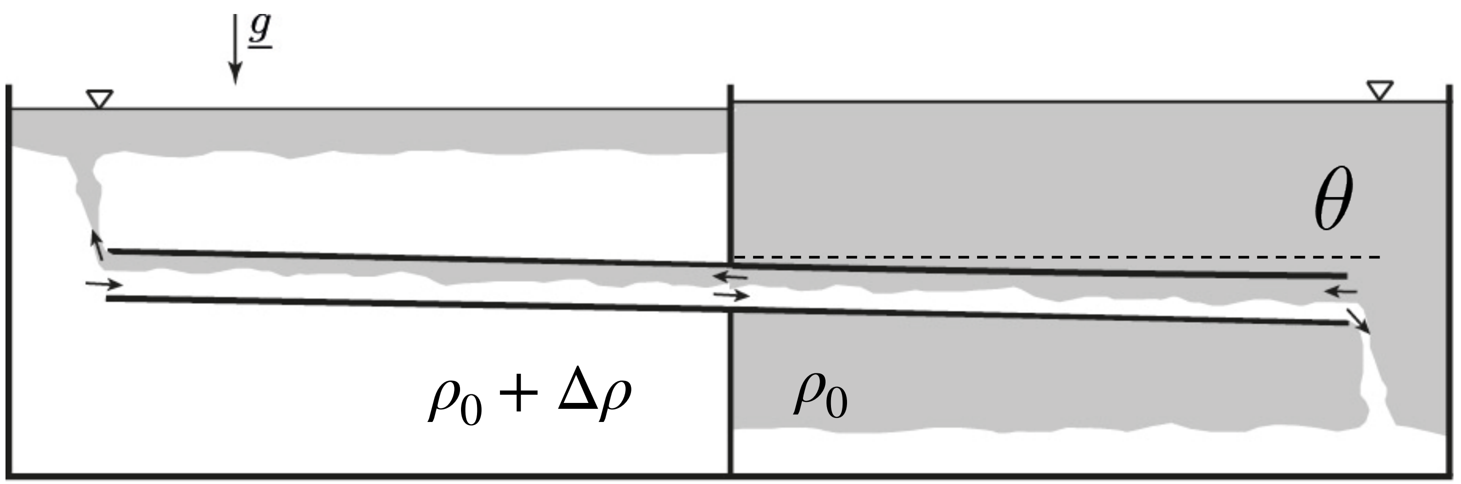
erc – European Research Council

UNIVERSITY OF CAMBRIDGE

The Stratified Inclined Duct (SID)

Meyer & Linden (2014)

- Exchange flow between two reservoirs
- Two-layer **stratified shear flow** with **sustained forcing**



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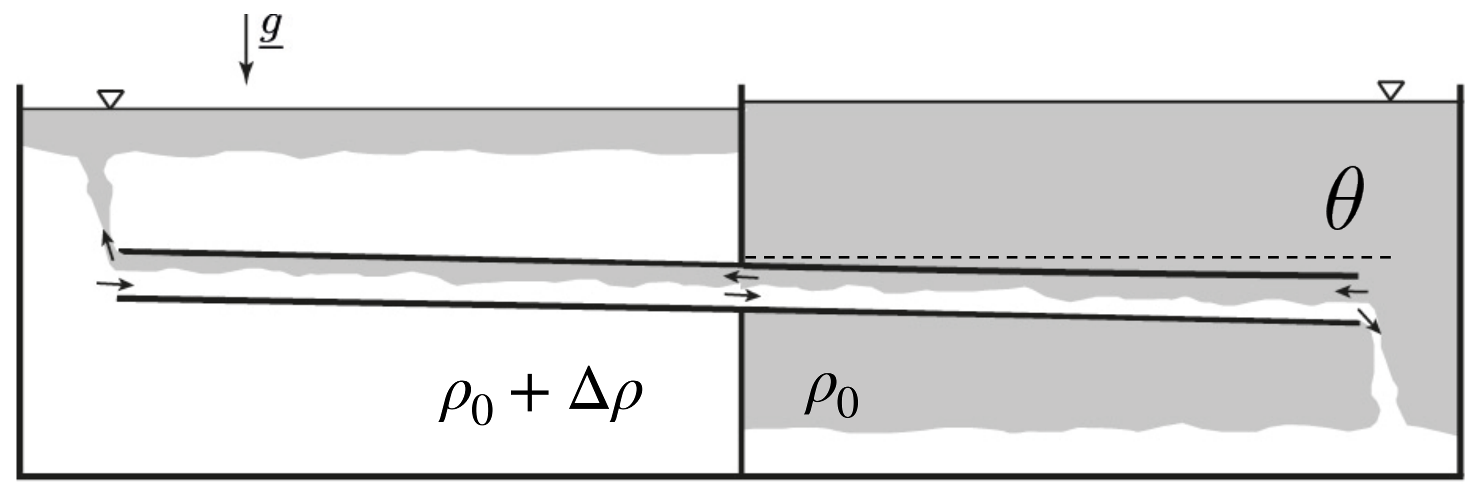

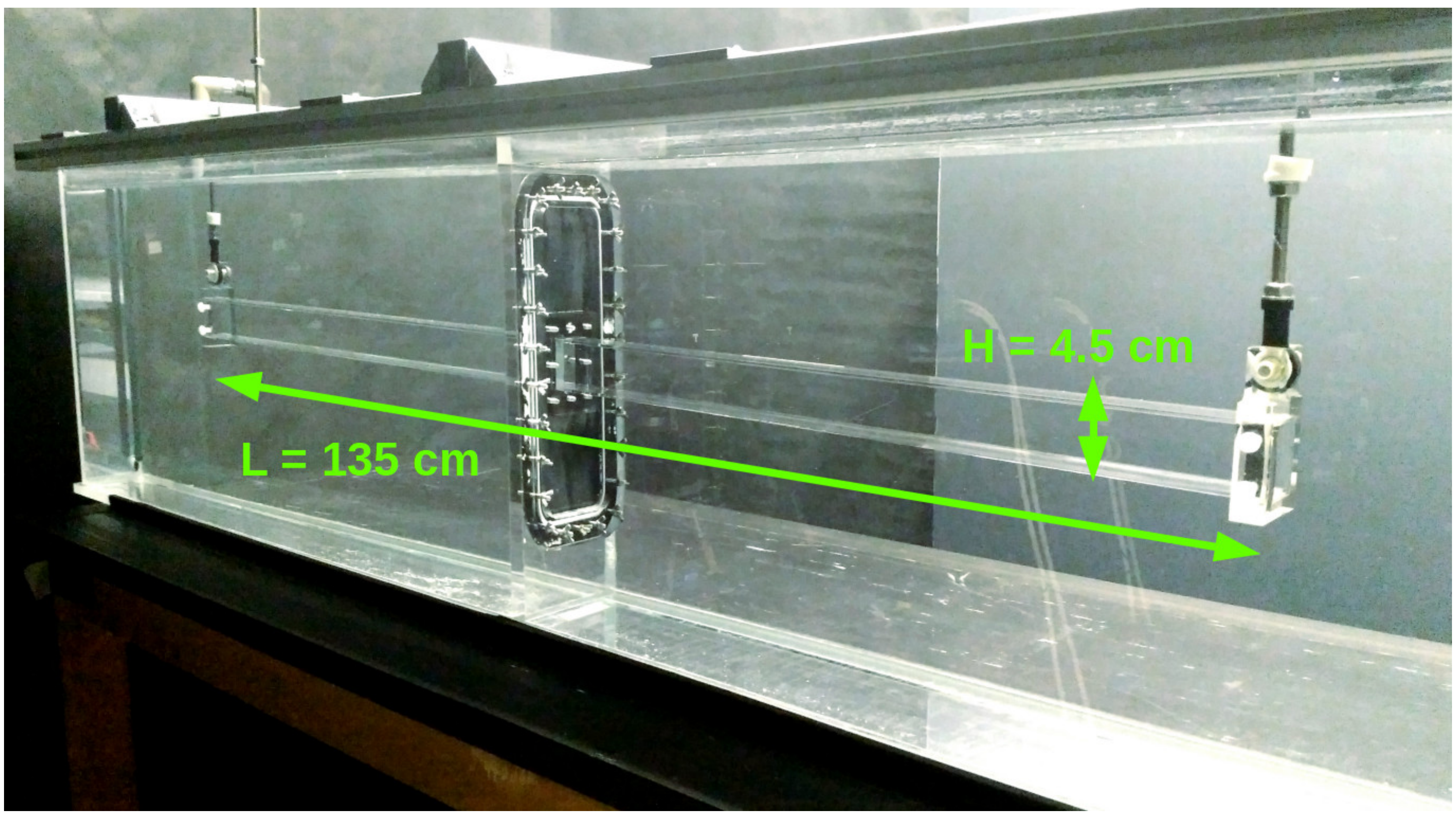

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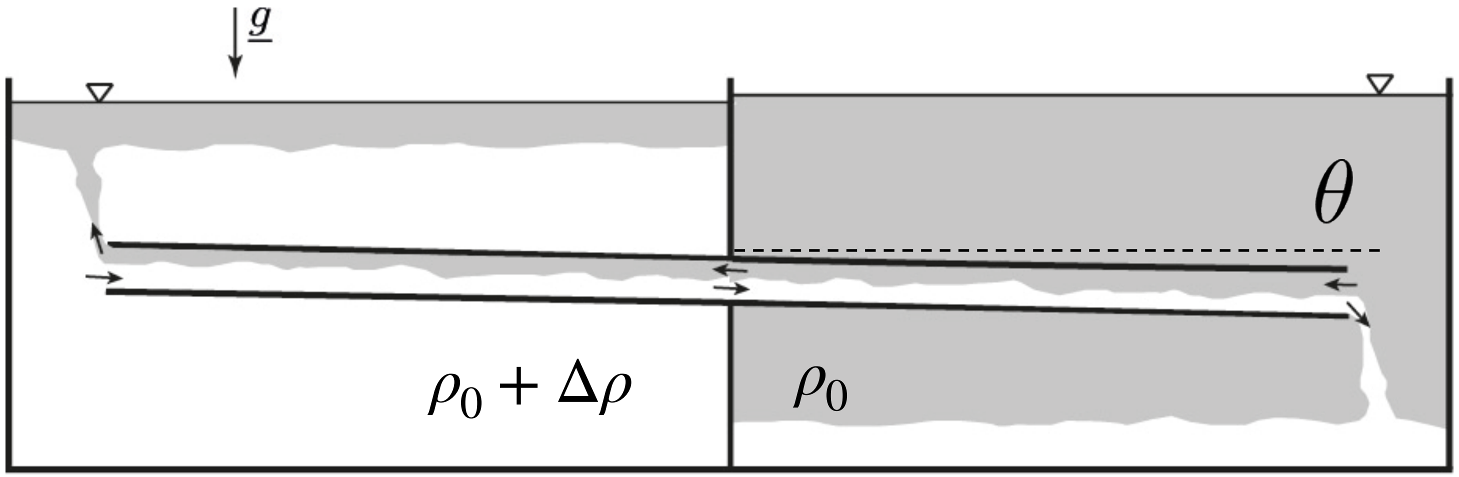

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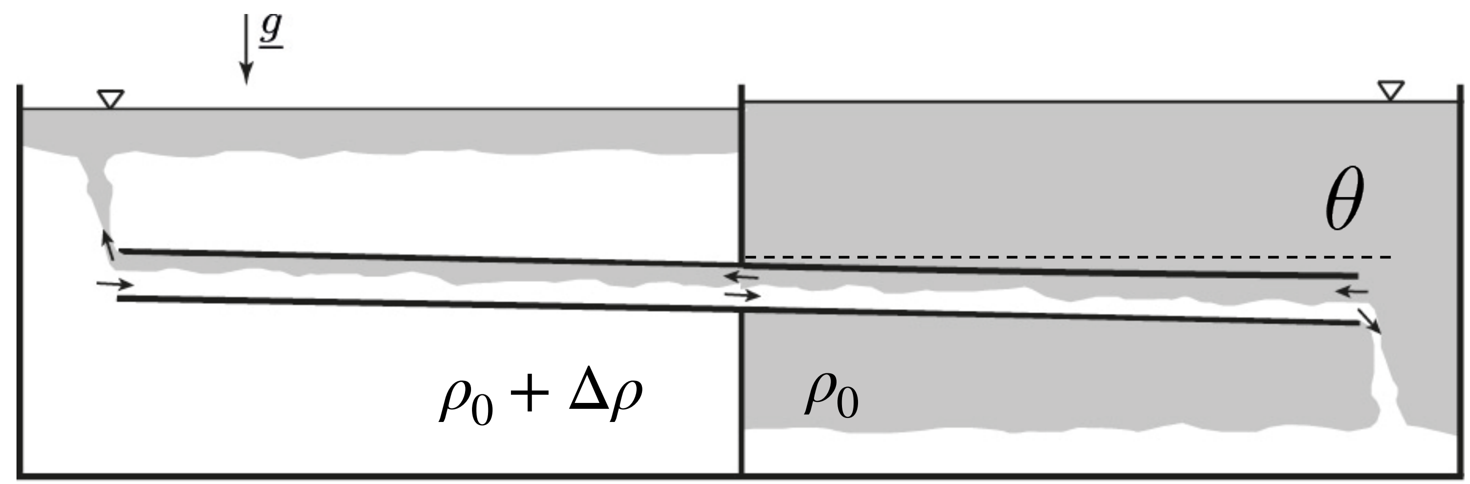
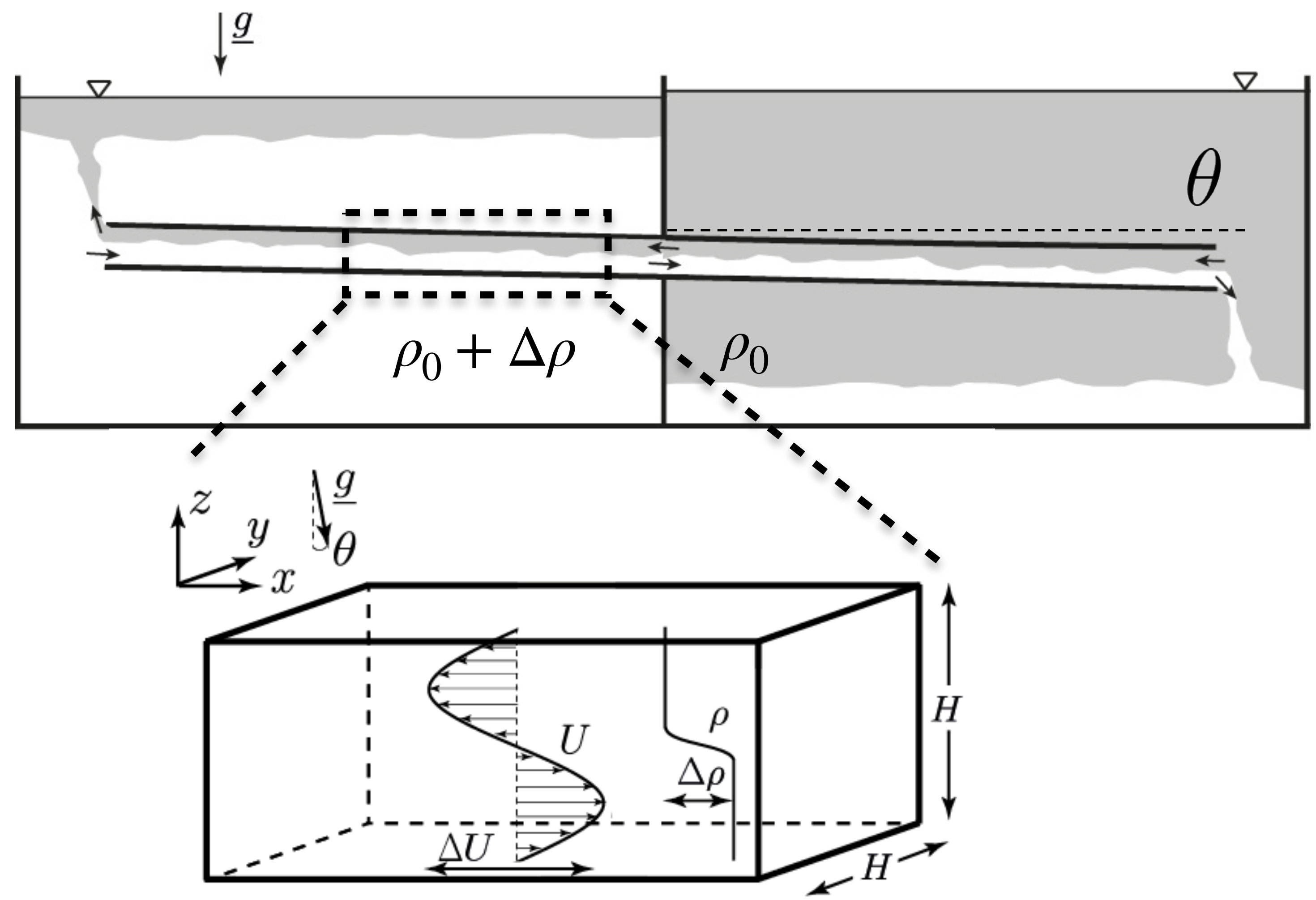
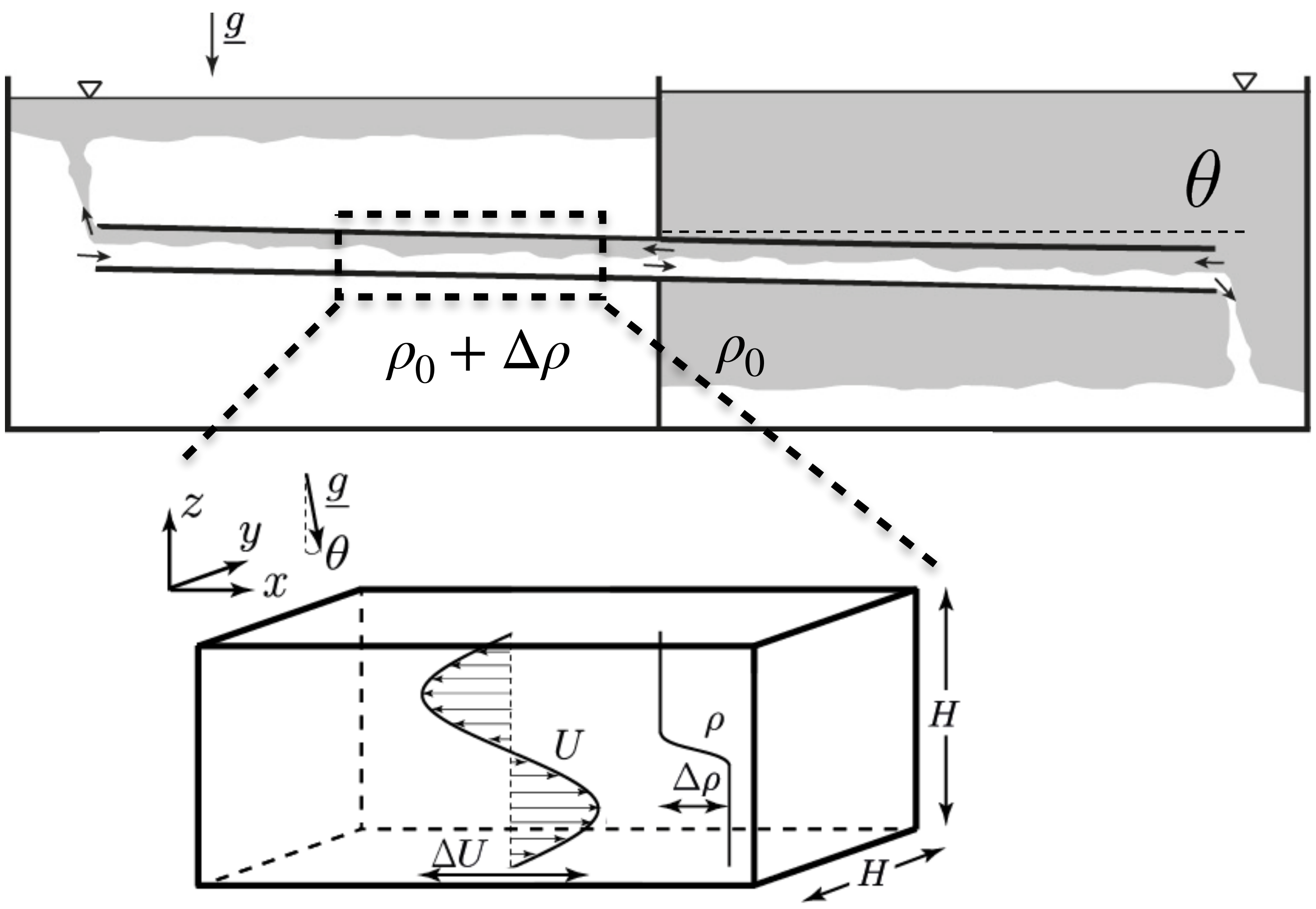




$$\Delta U = \sqrt{g'H}$$



$$\Delta U = \sqrt{g'H}$$

$$Re = \frac{\sqrt{g'H}H}{2\nu} = 1.4 \times 10^4 \sqrt{\frac{\Delta\rho}{\rho_0}}$$

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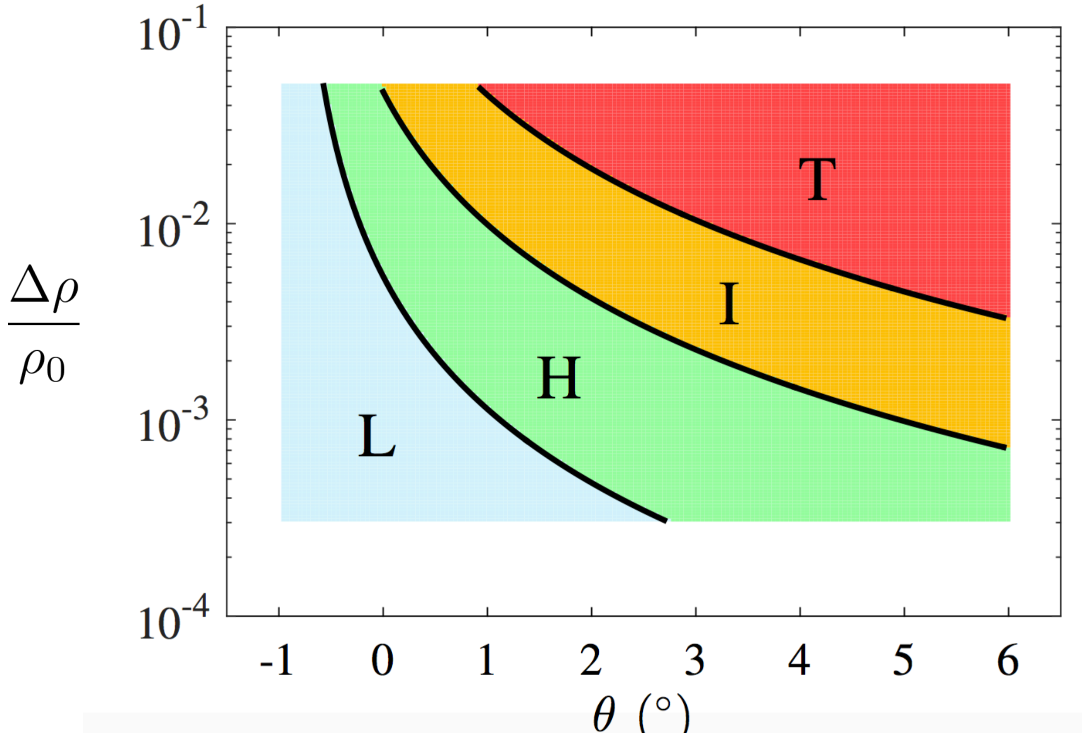

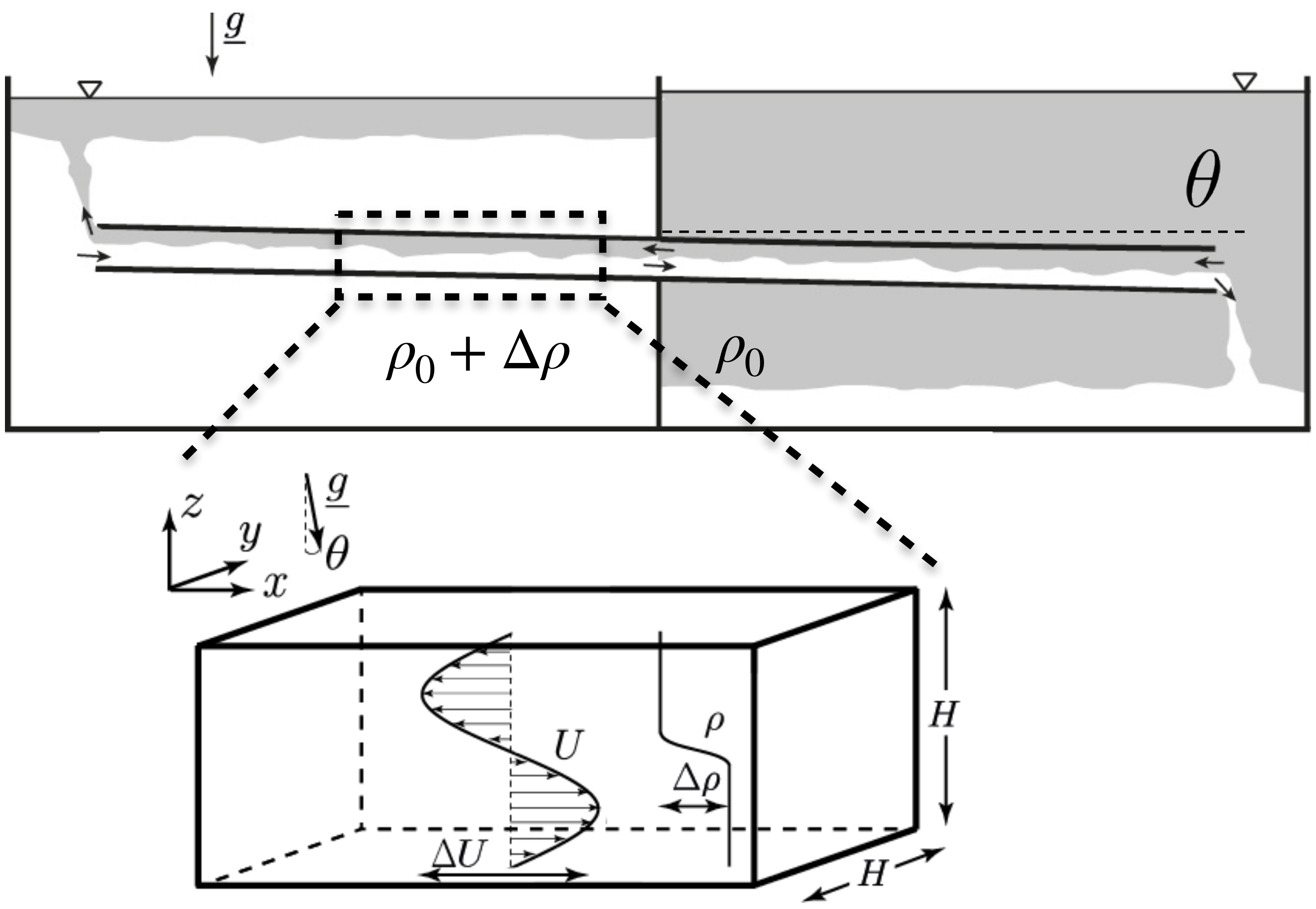

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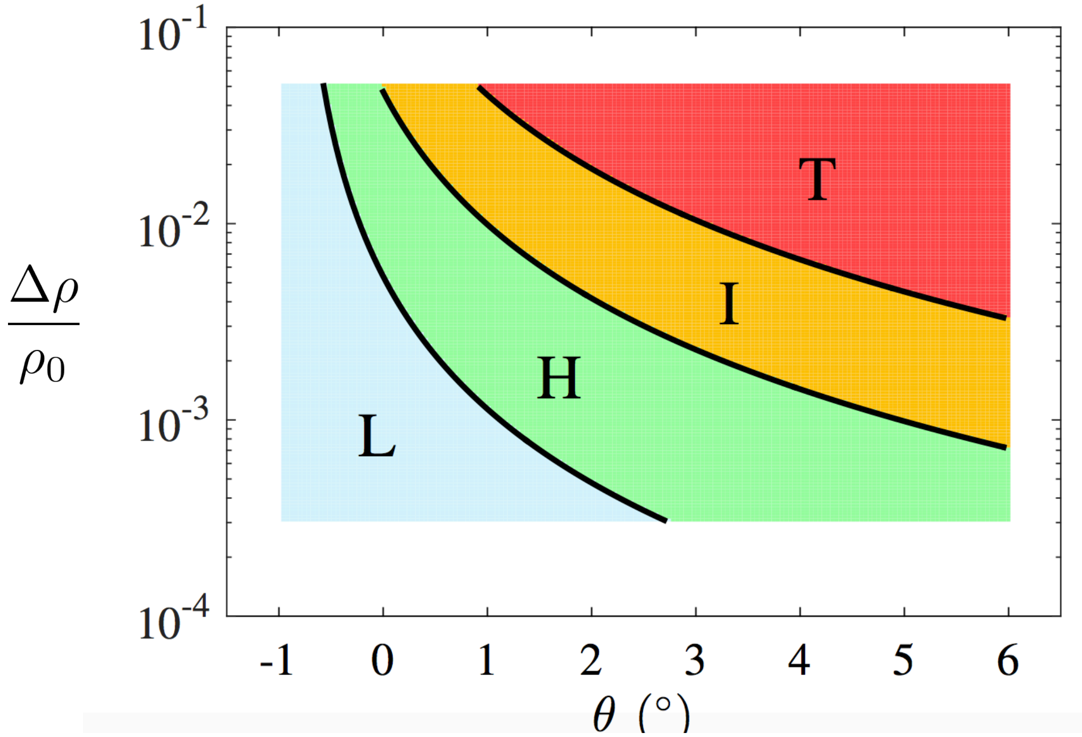

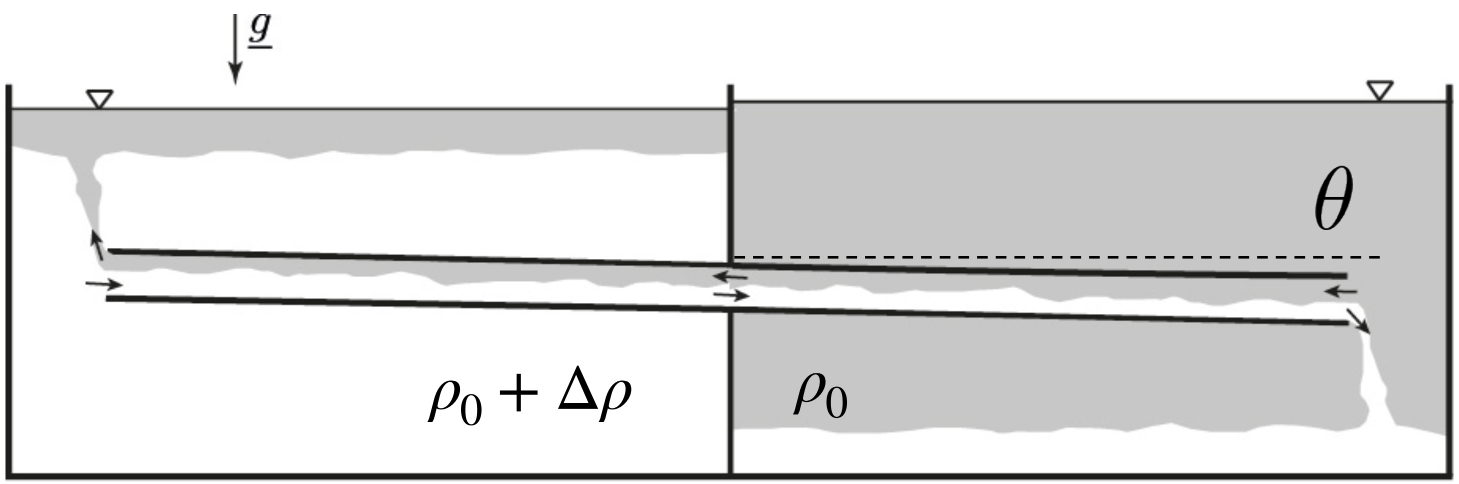

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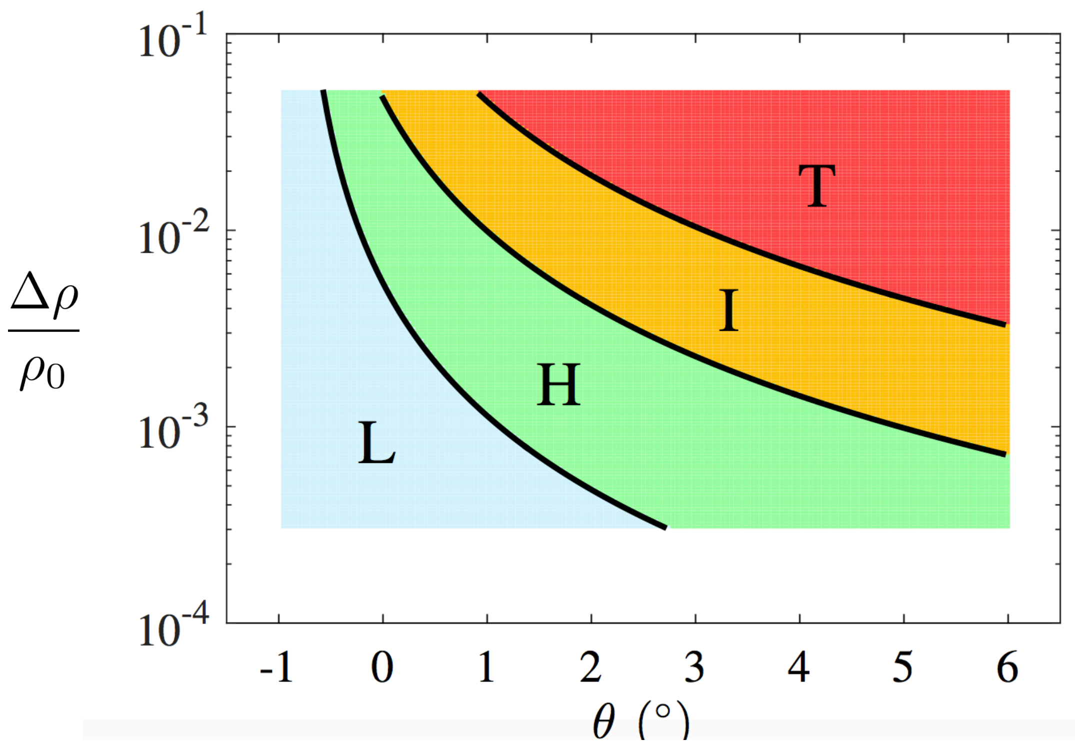

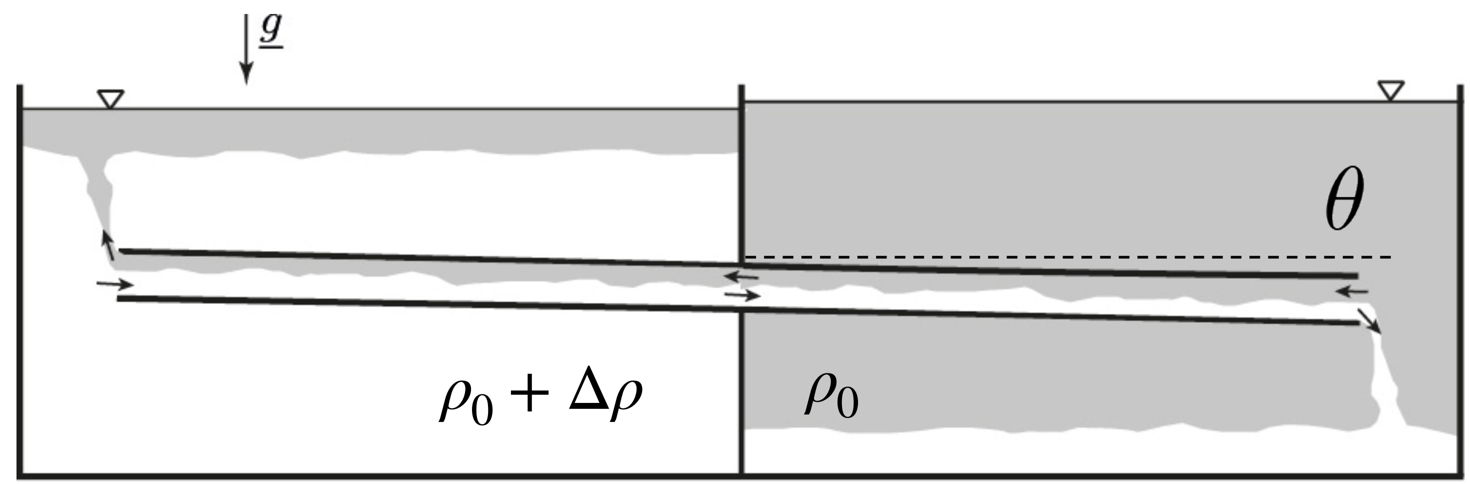

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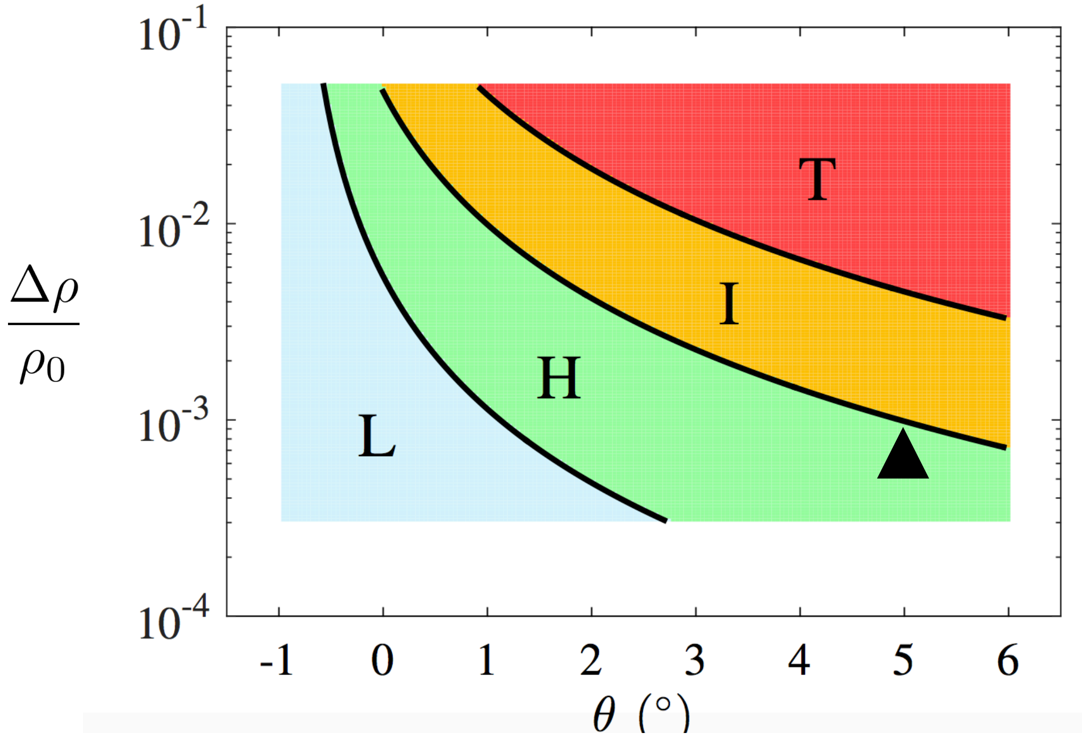

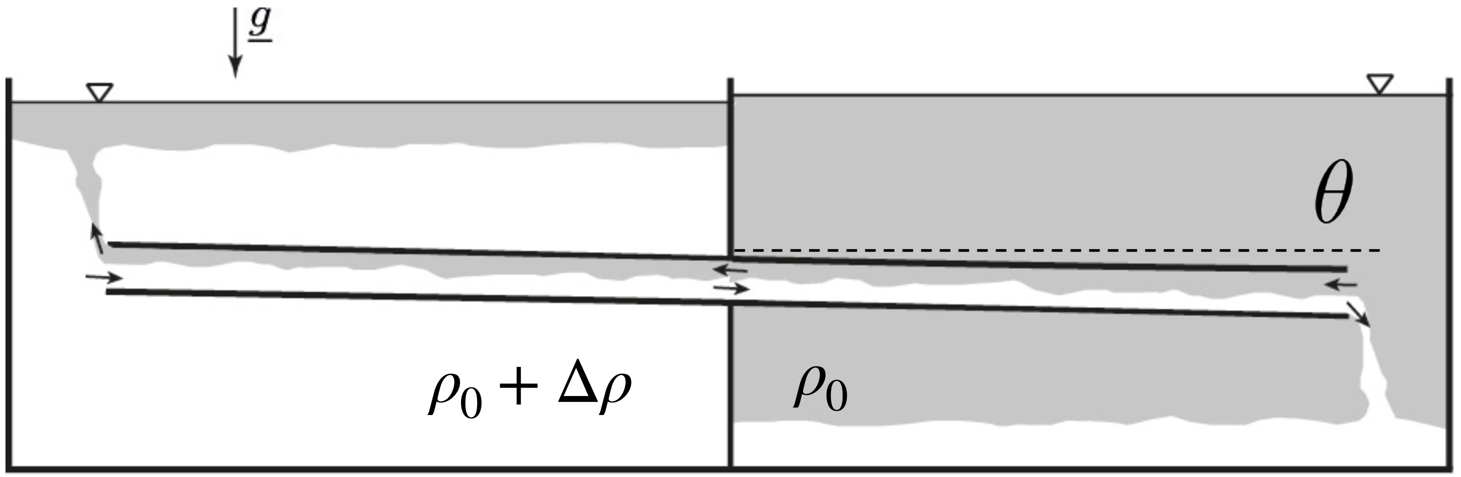

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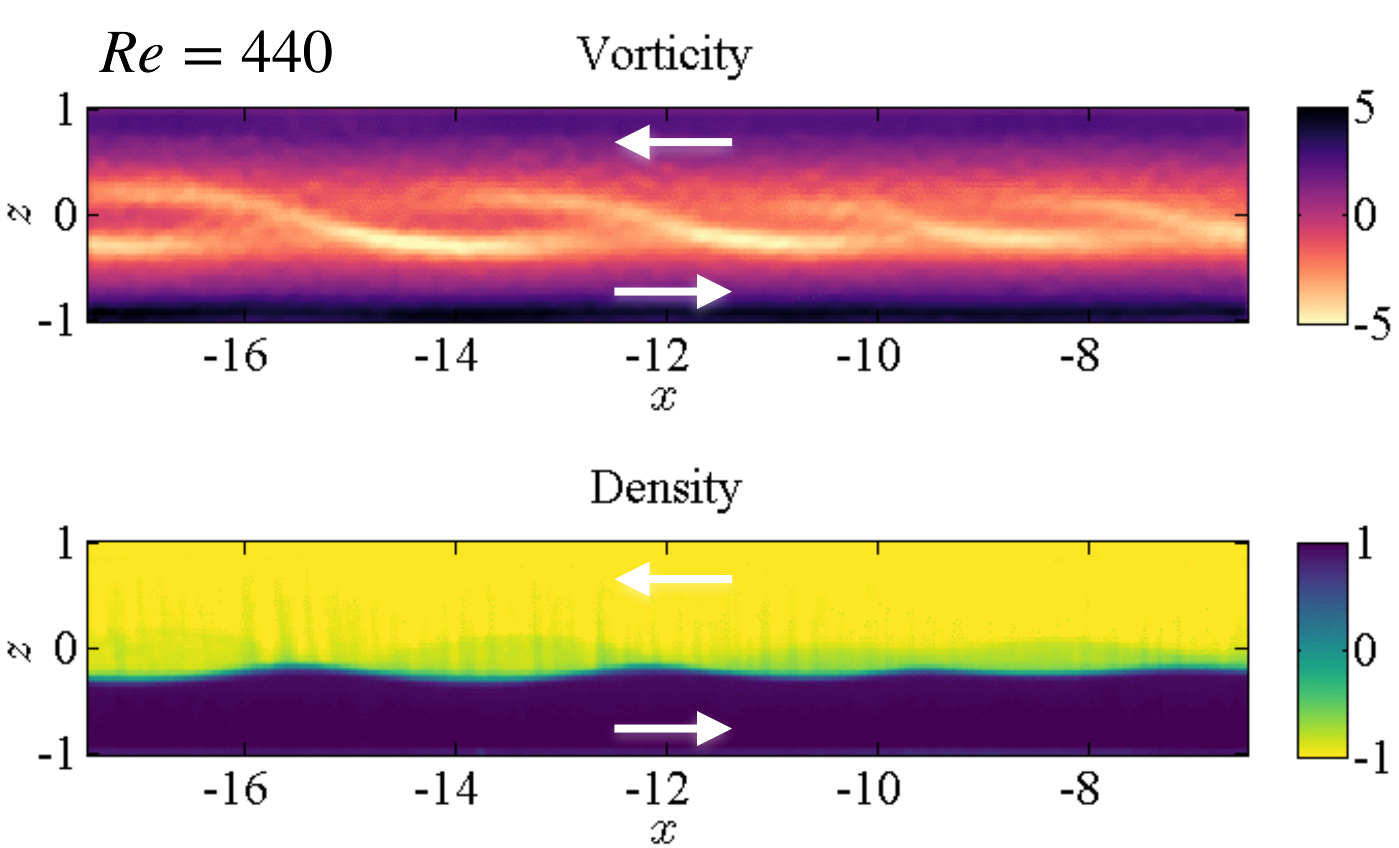

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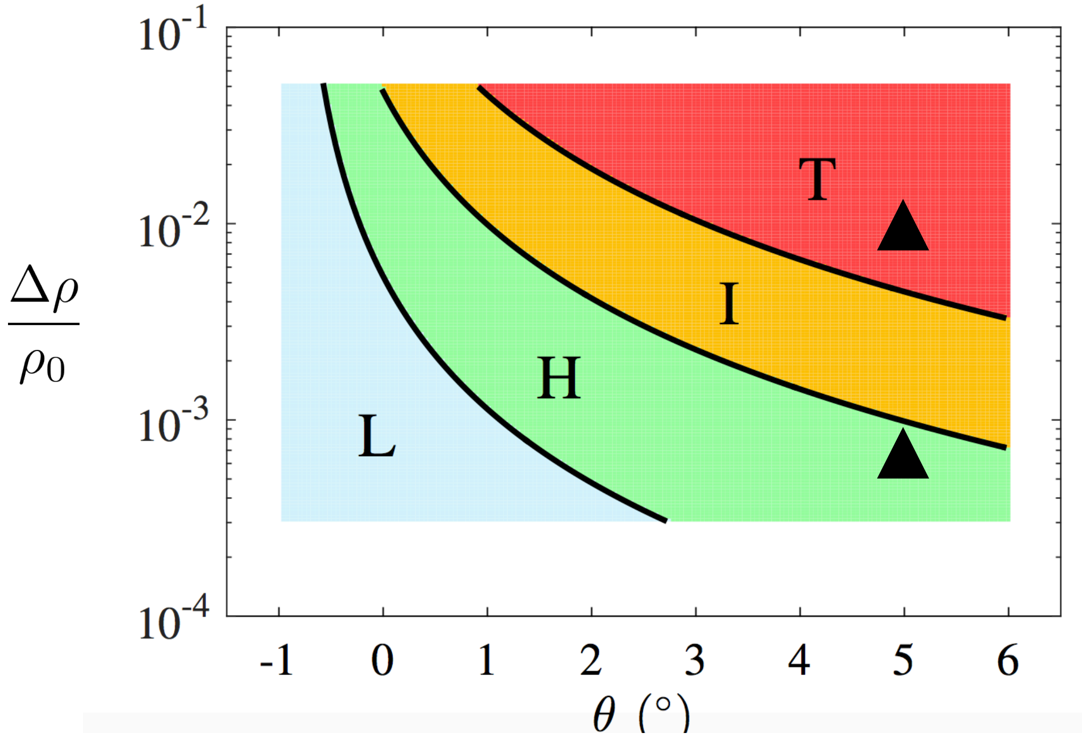

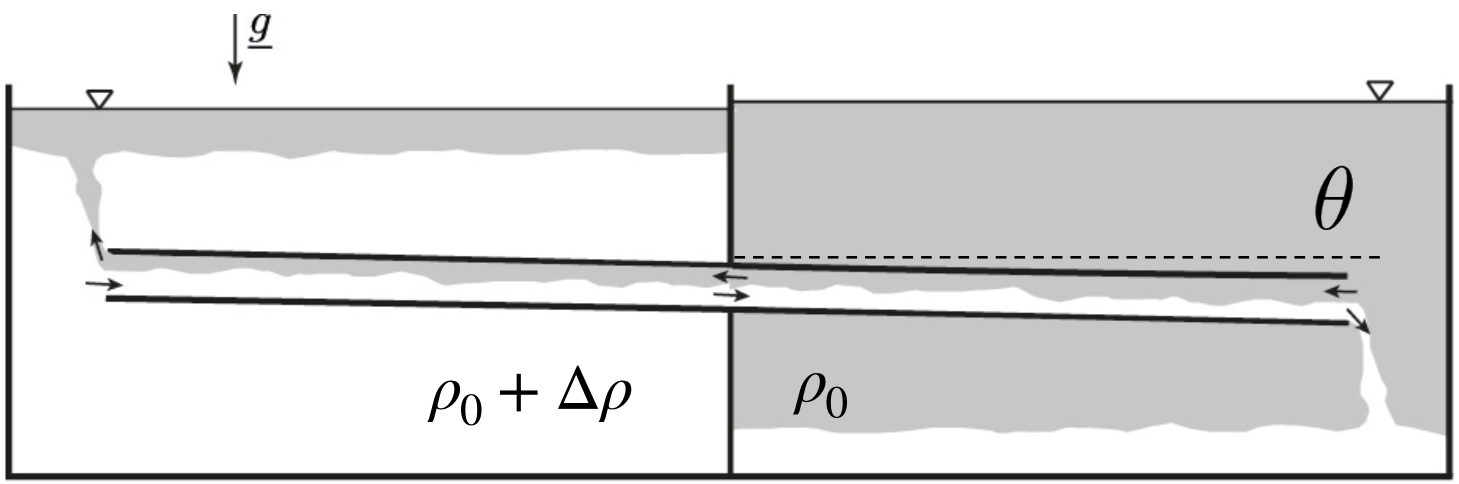

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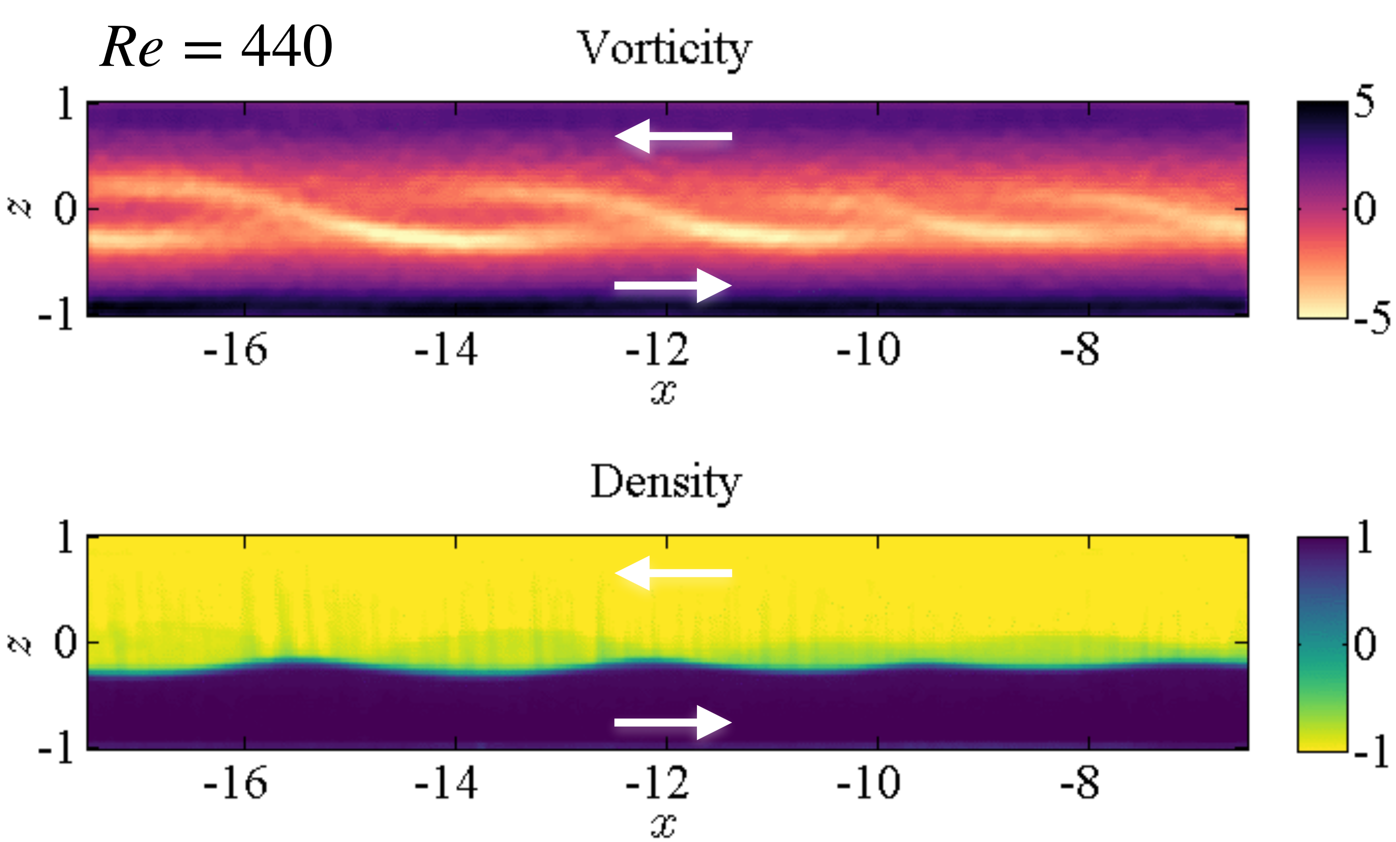

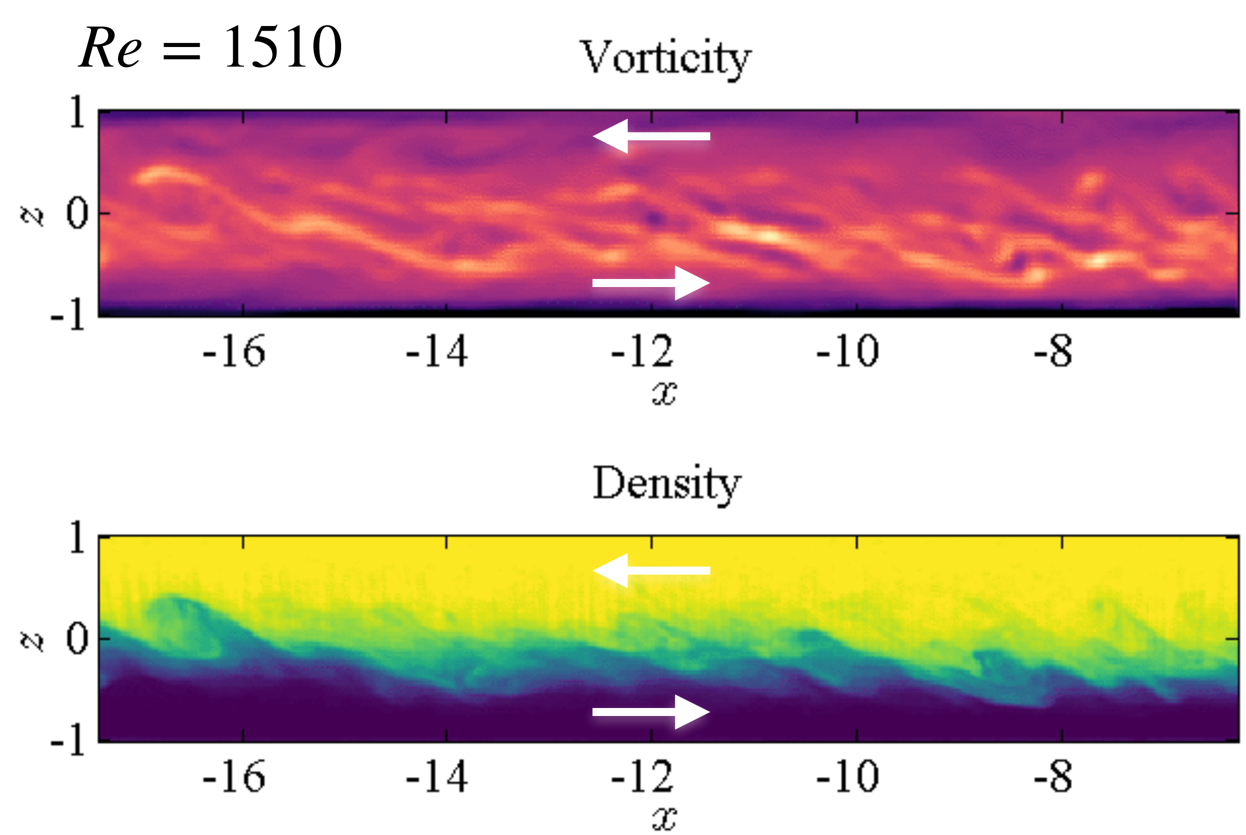

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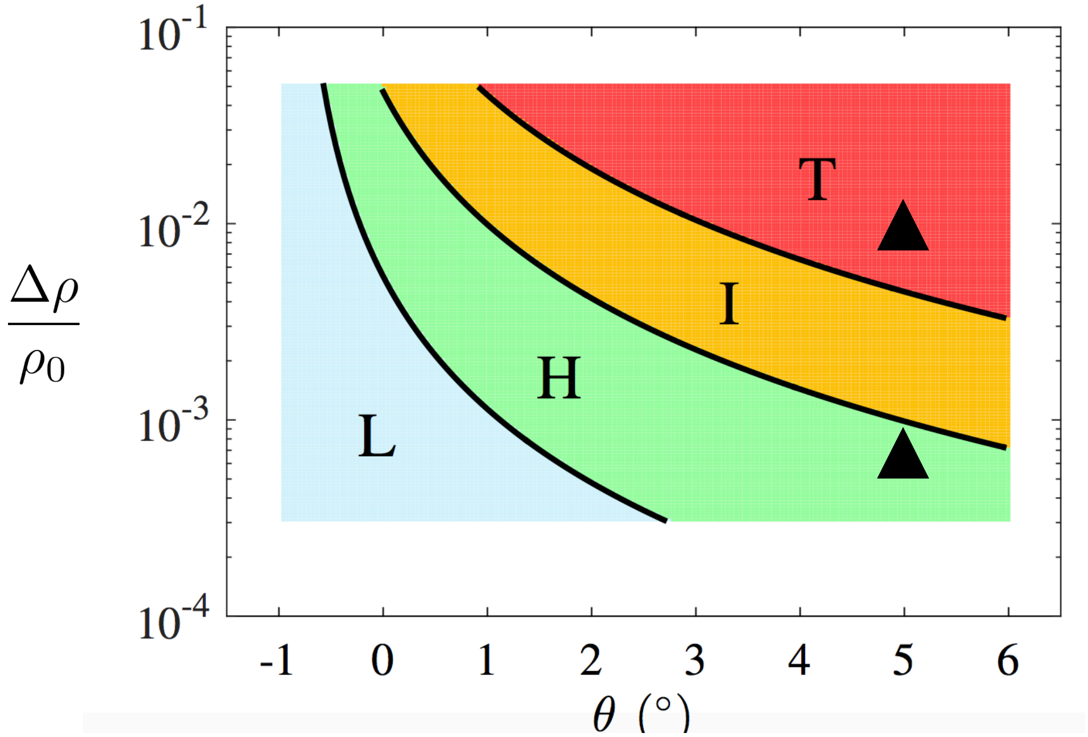

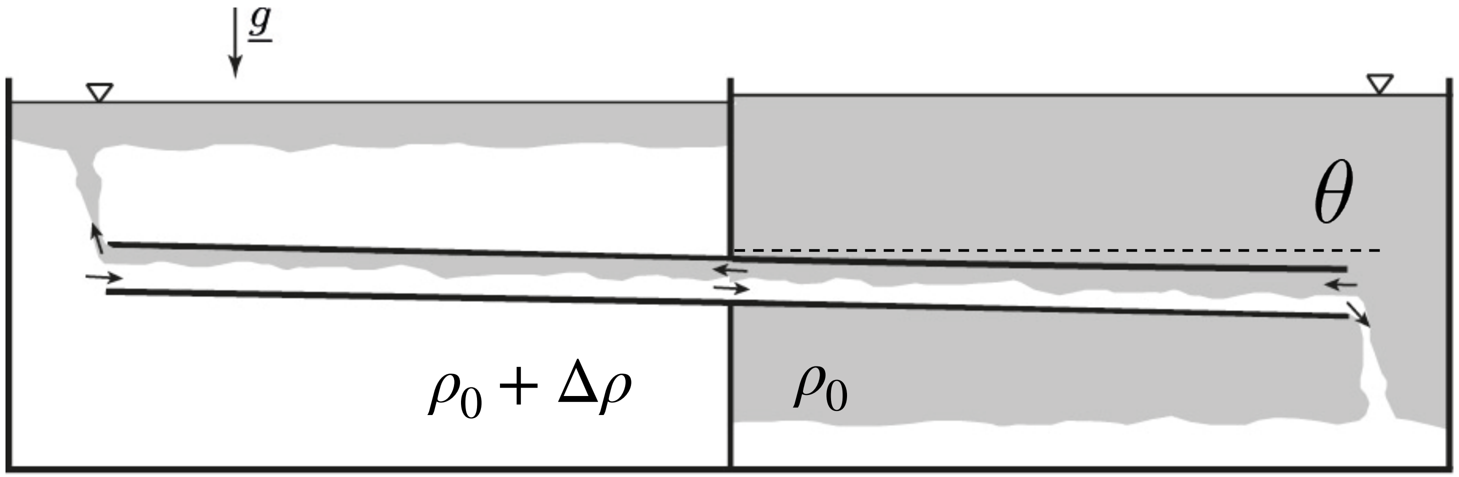

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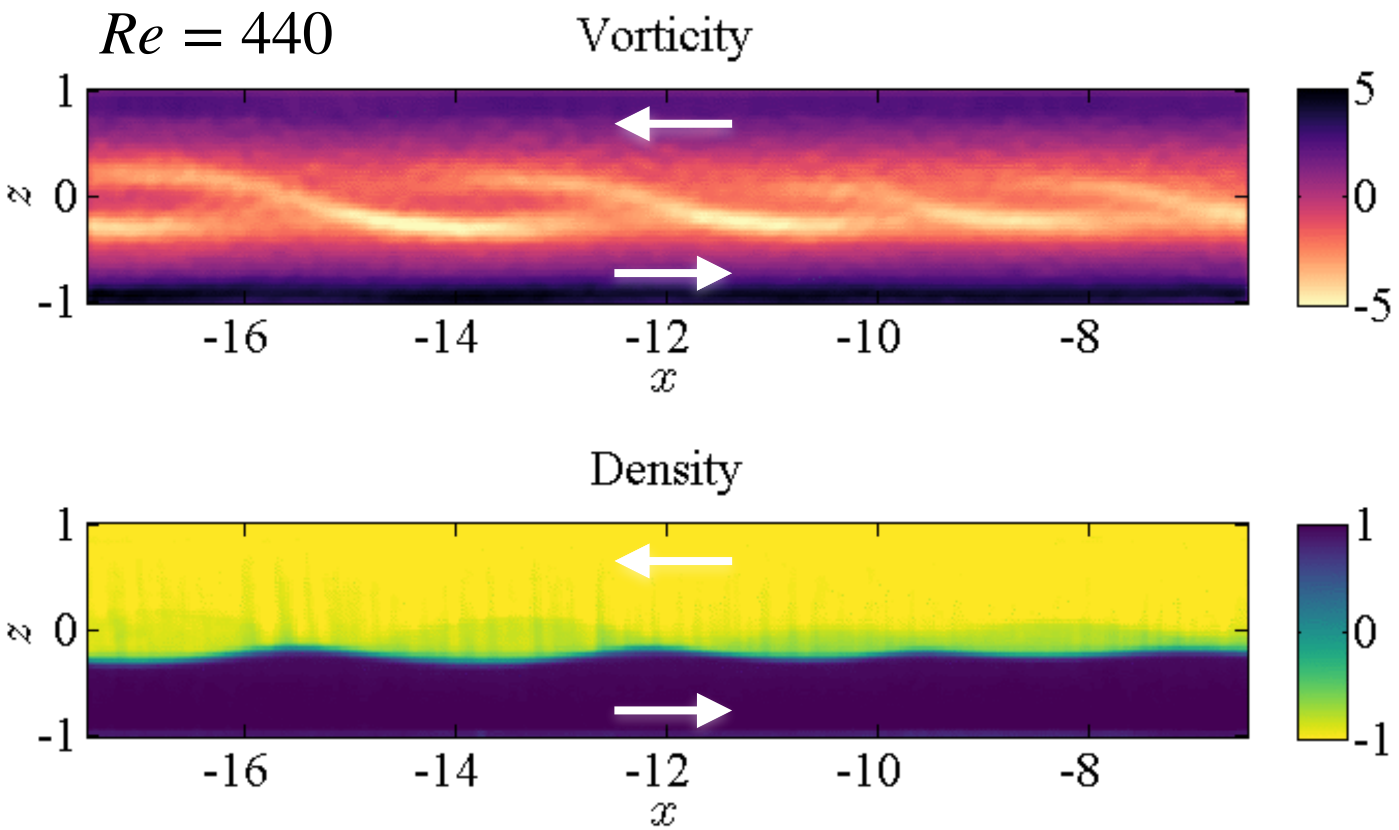

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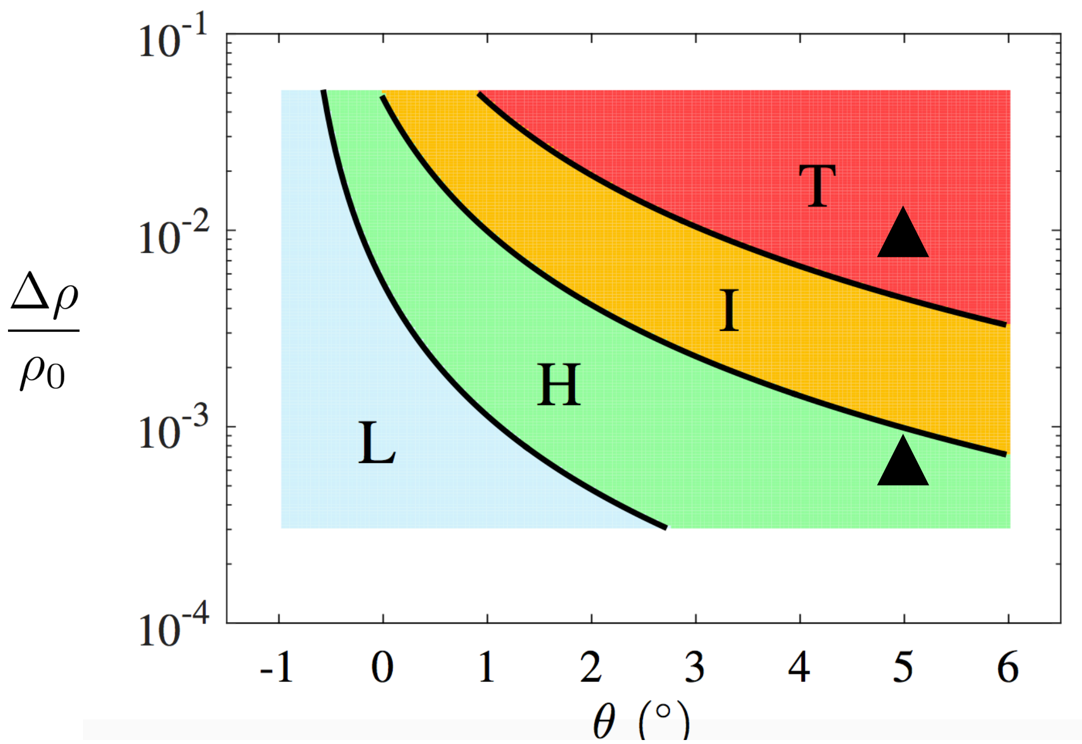

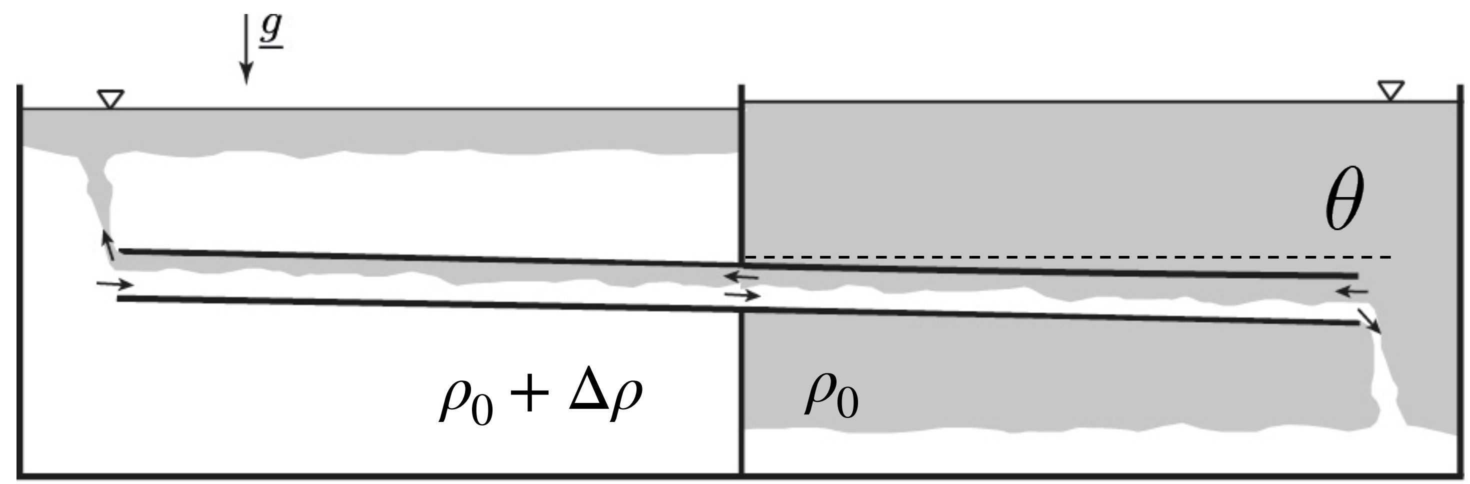

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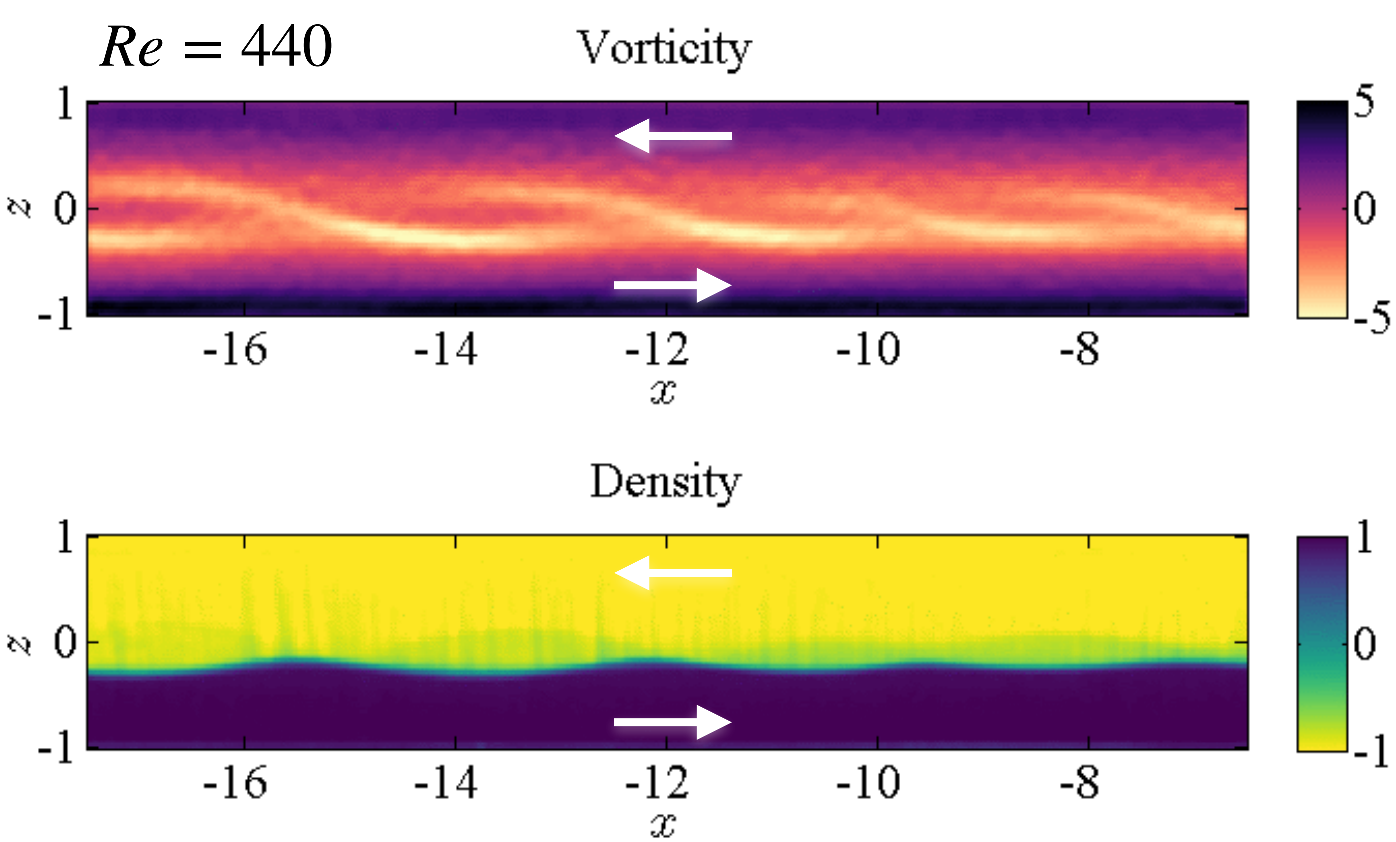


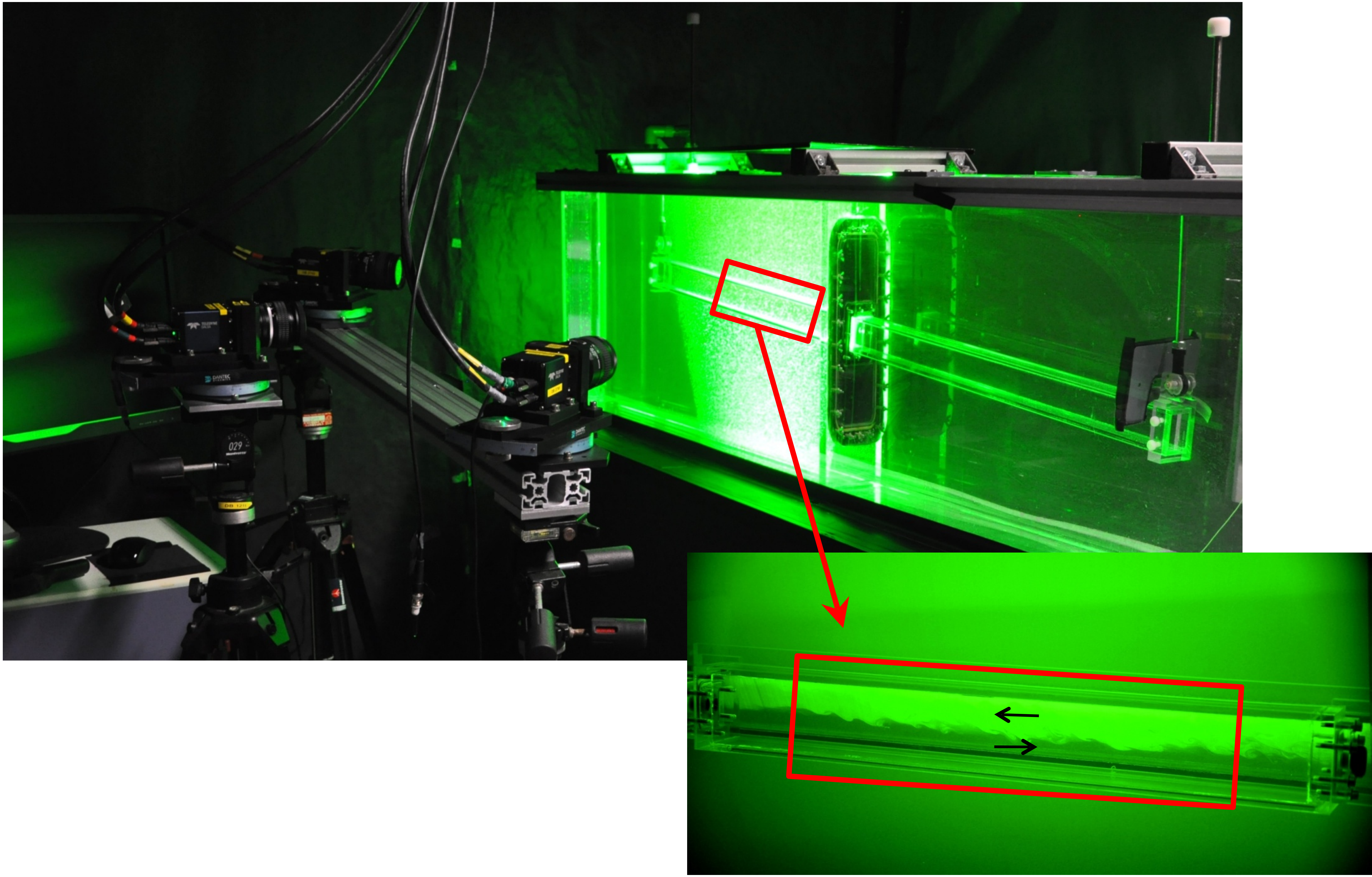
In this talk:

What is this low-Re coherent structure?

- **What 3D structure?**
- **What physical mechanism?**

Volumetric measurements

Partridge, Lefauve & Dalziel (2019)



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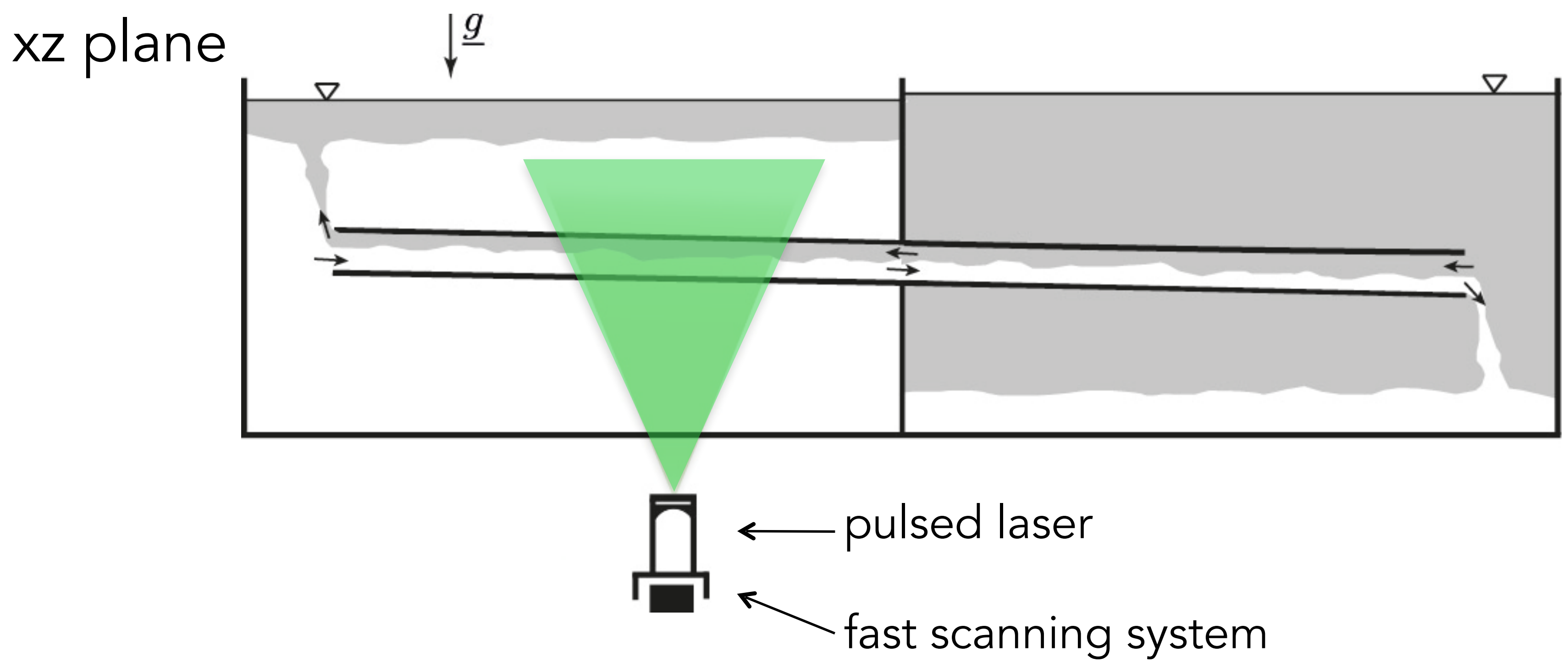

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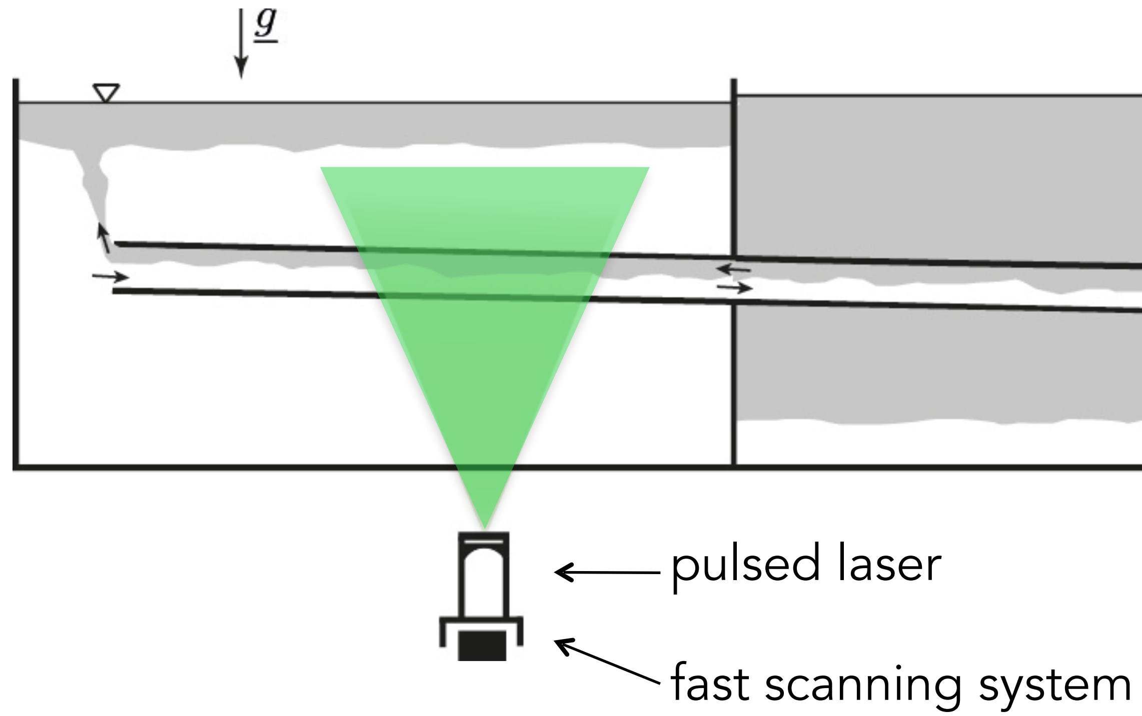
xz plane



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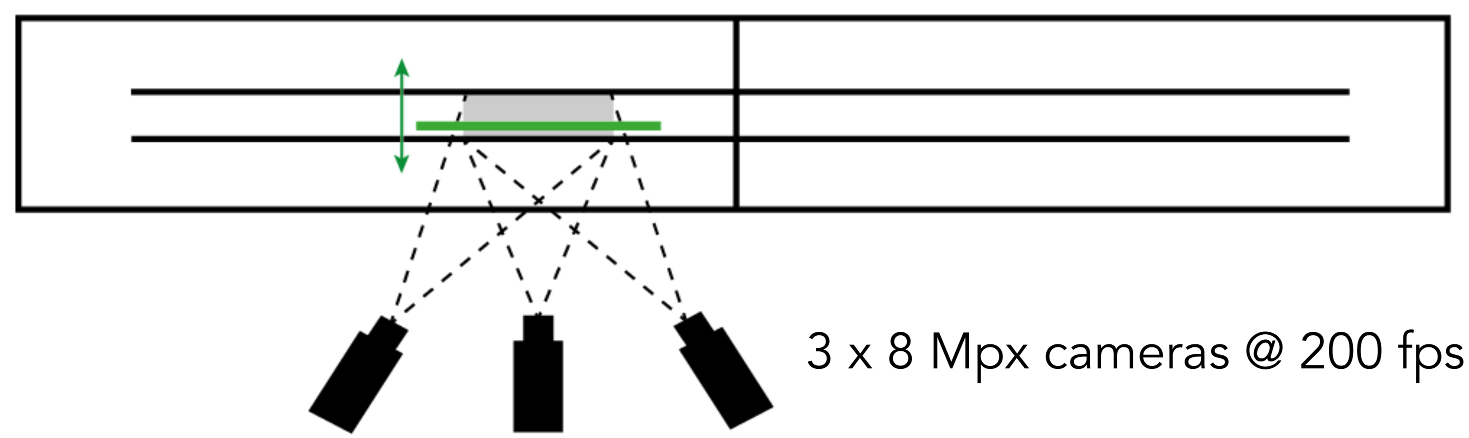

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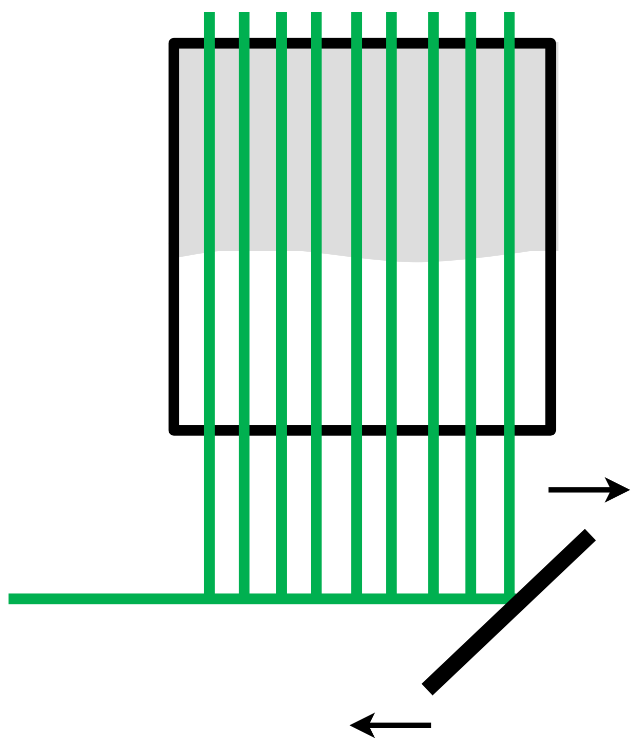

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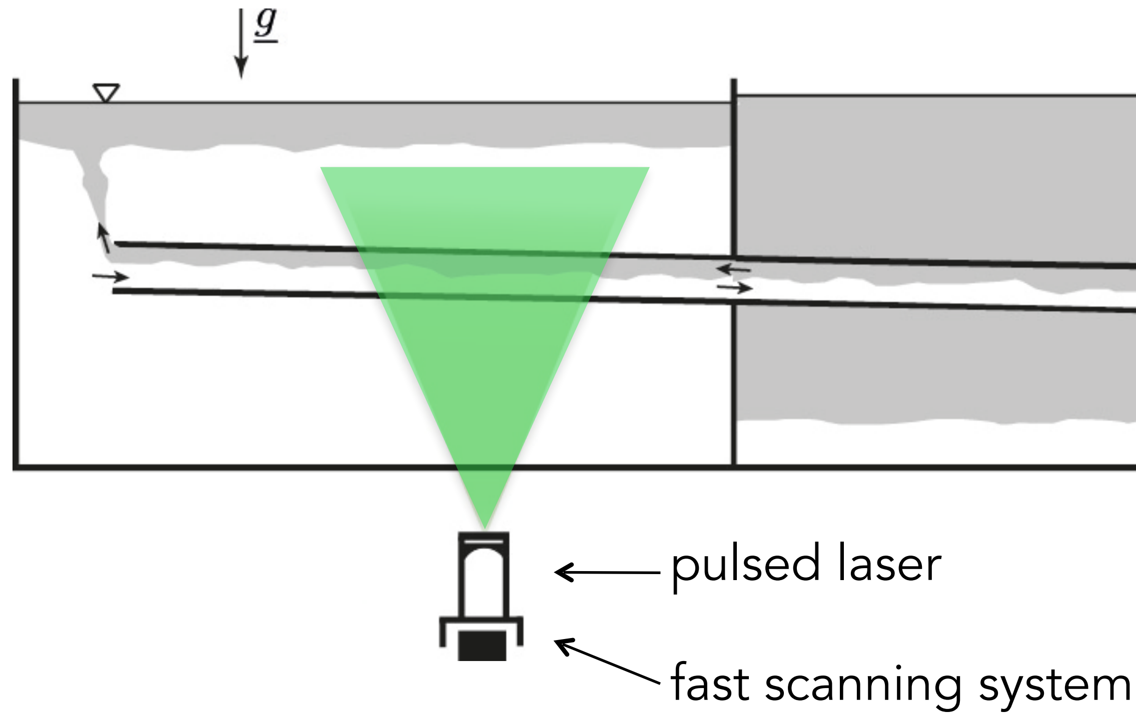

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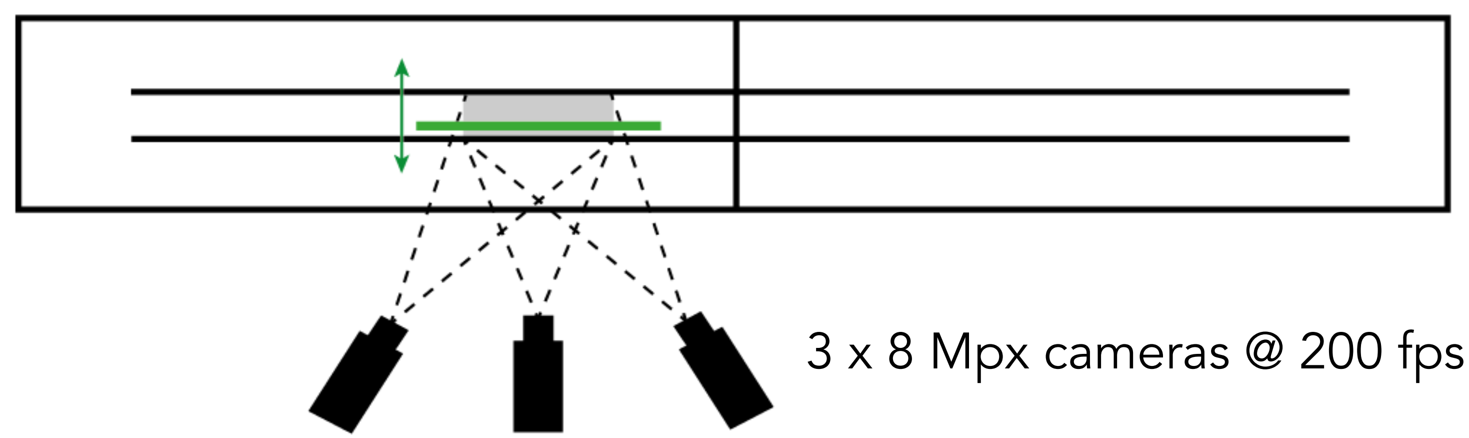

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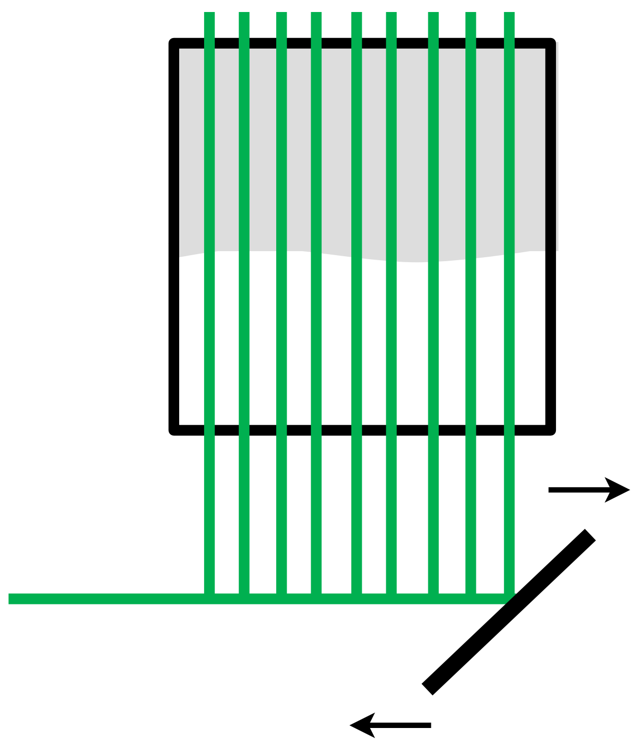

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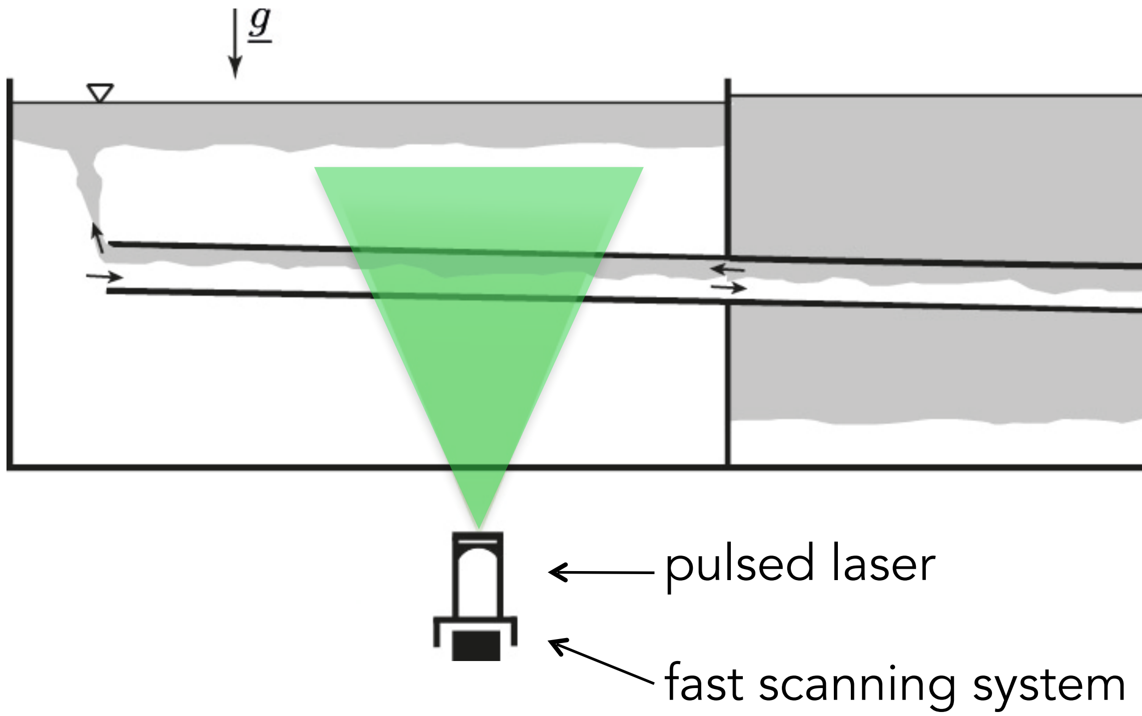

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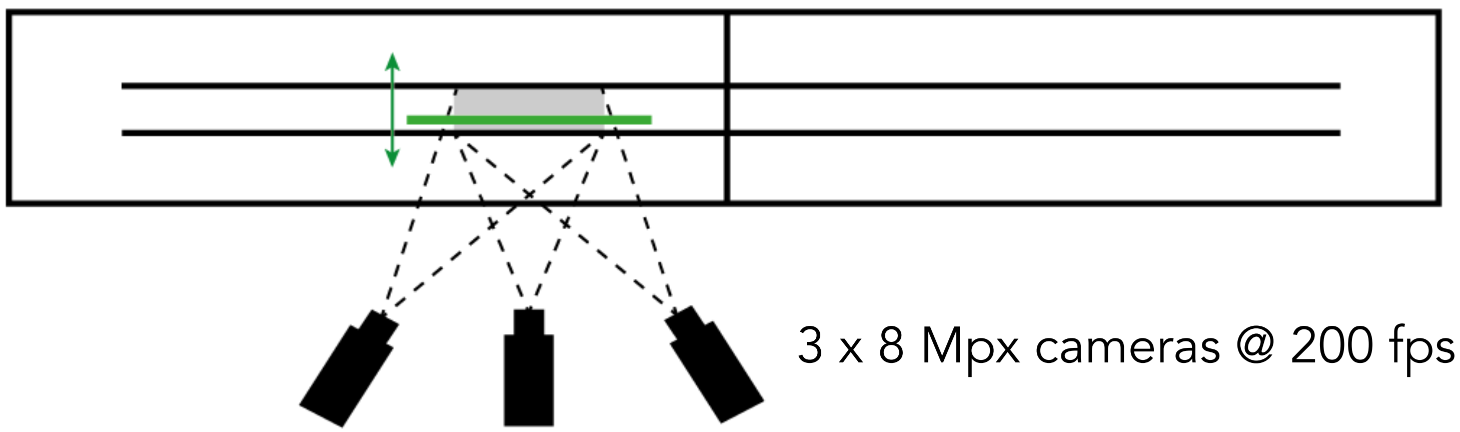

Stereo **P**article **I**mage **V**elocimetry

+

Planar **L**aser **I**nduced **F**luorescence

$\longrightarrow u, v, w, \rho\,(x, y_i, z, t_i)$

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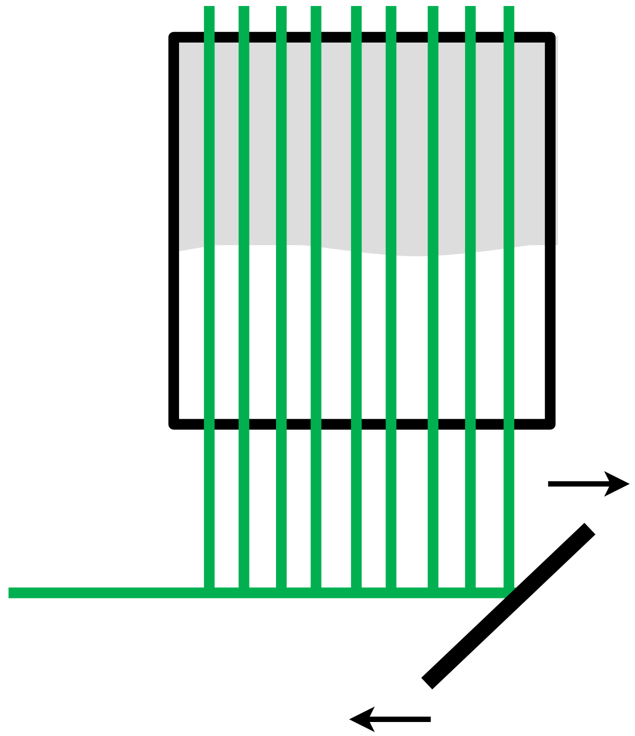

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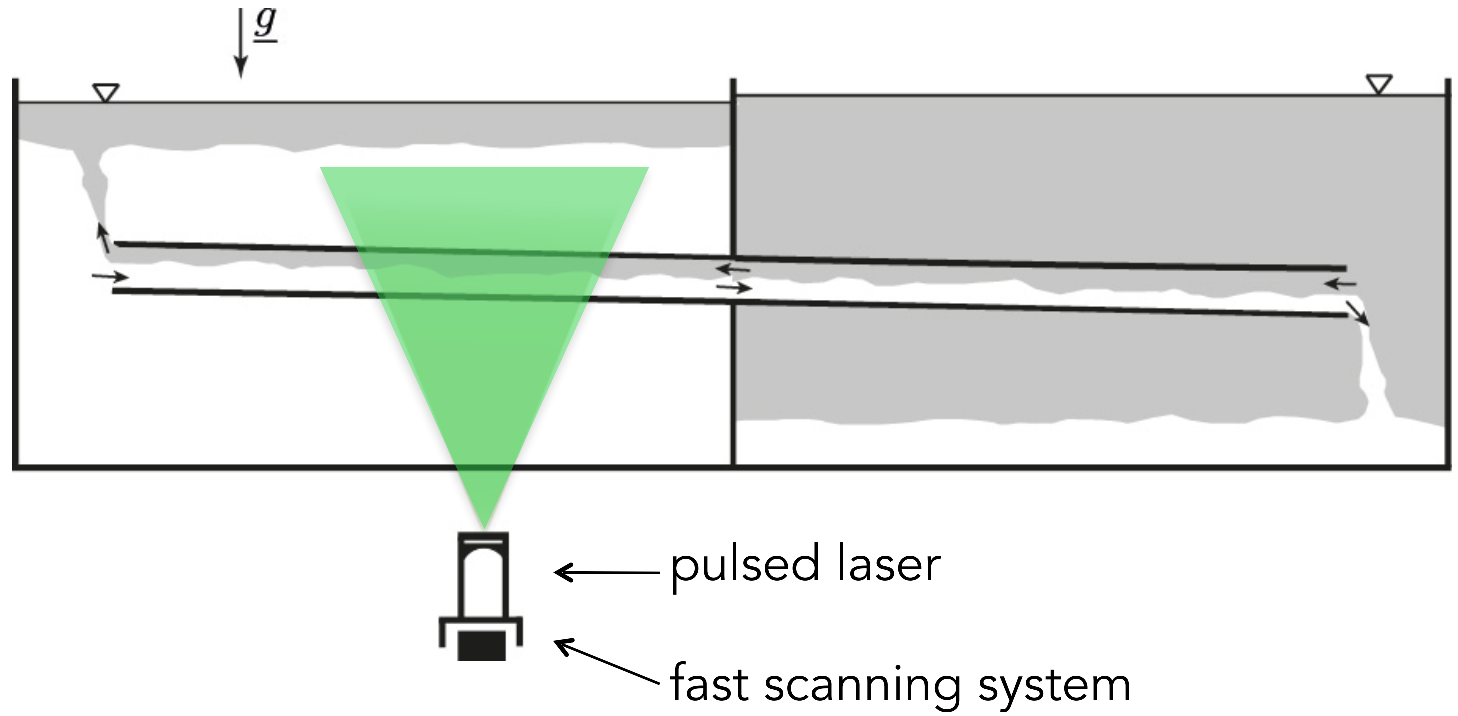

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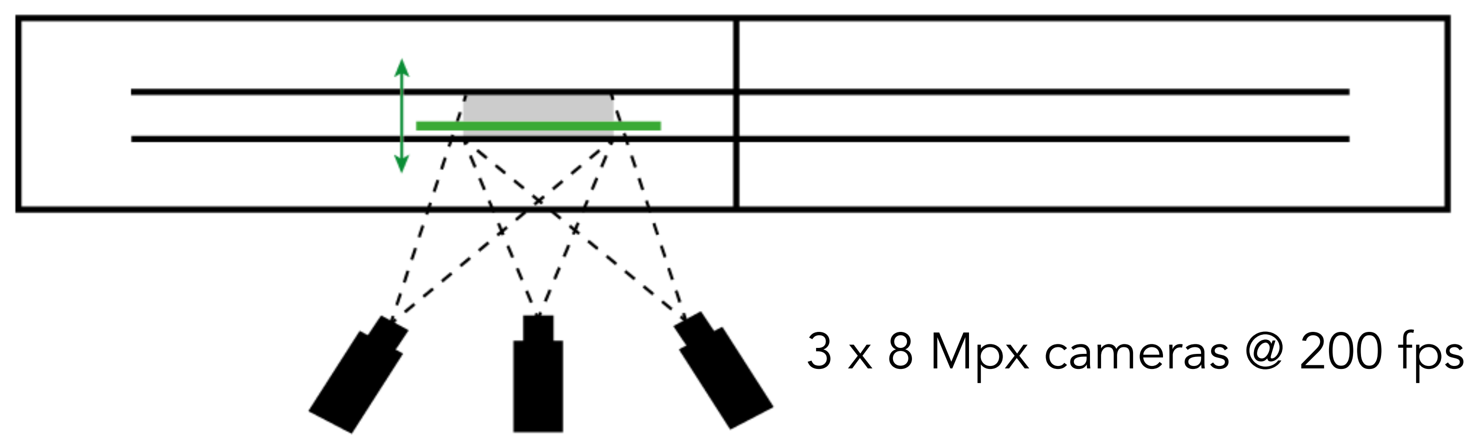

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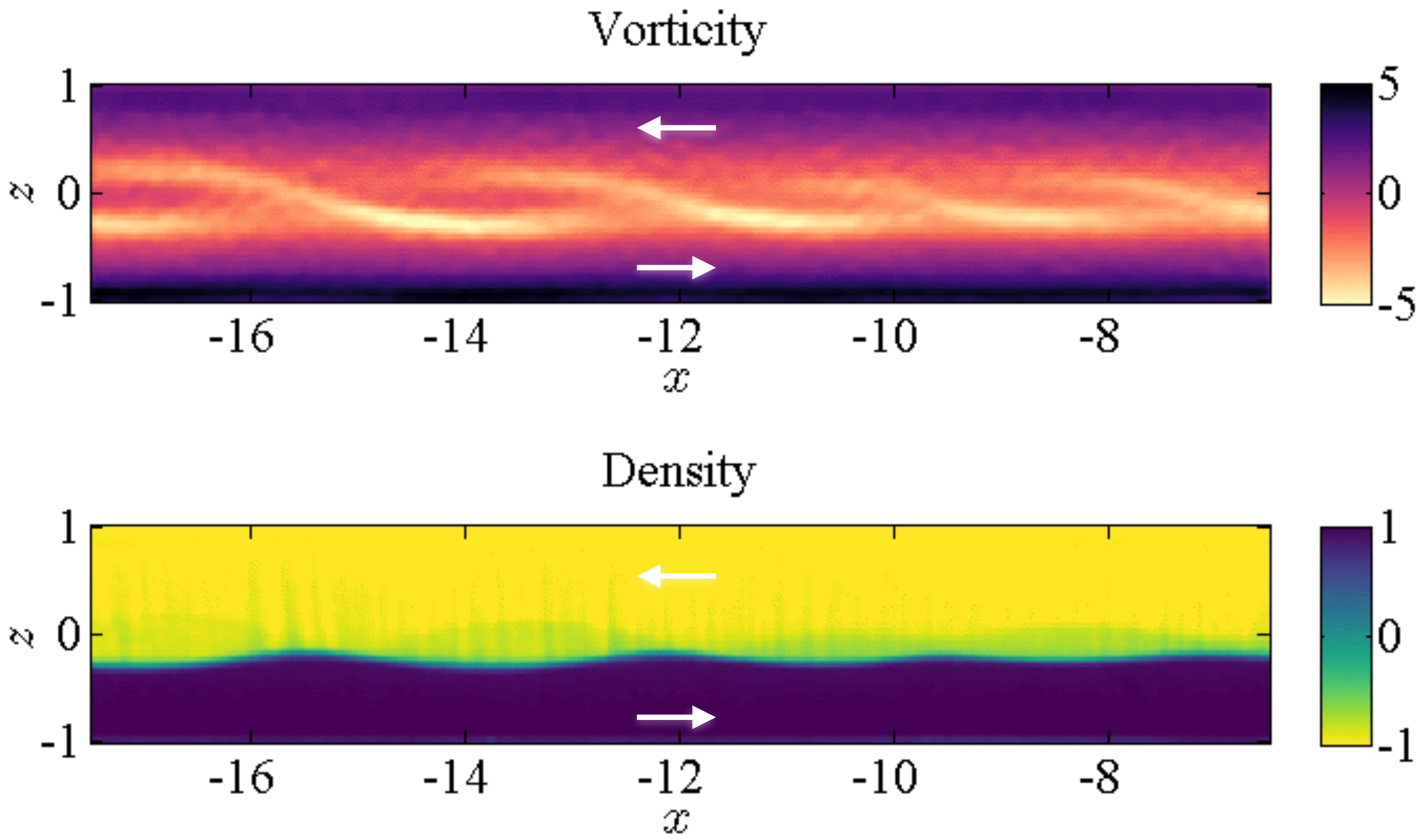
$\longrightarrow u, v, w, \rho\,(x, y_i, z, t_i)$

in $i = 1, \ldots, 30$ successive planes $\longrightarrow$ construct 3D volumes $u, v, w, \rho\,(x, y, z, t)$

vector yield ~ $4 \times 500 \times 30 \times 100 \times 300 \sim 2 \times 10^9$ / experiment

3D structure: isosurfaces



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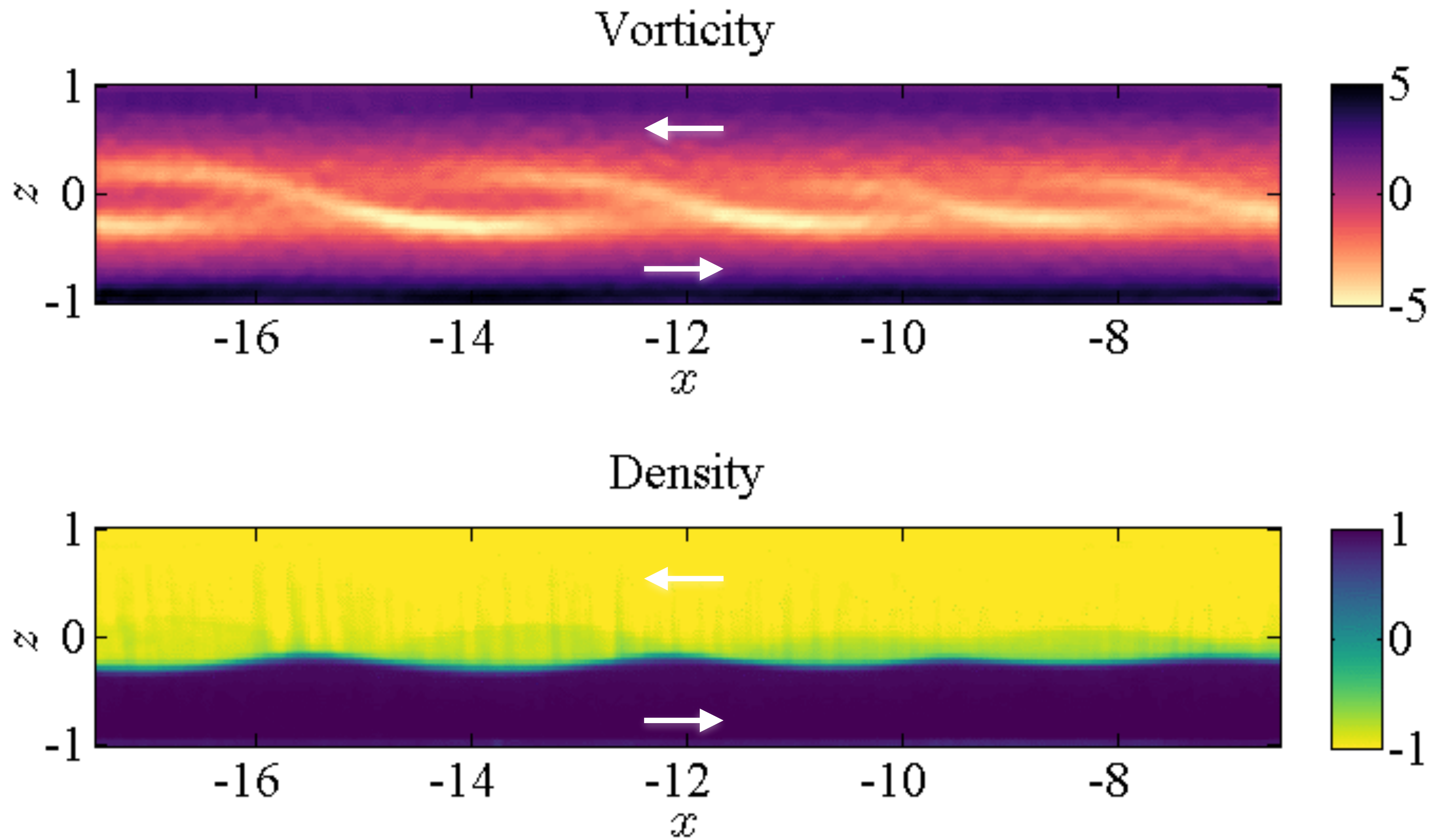

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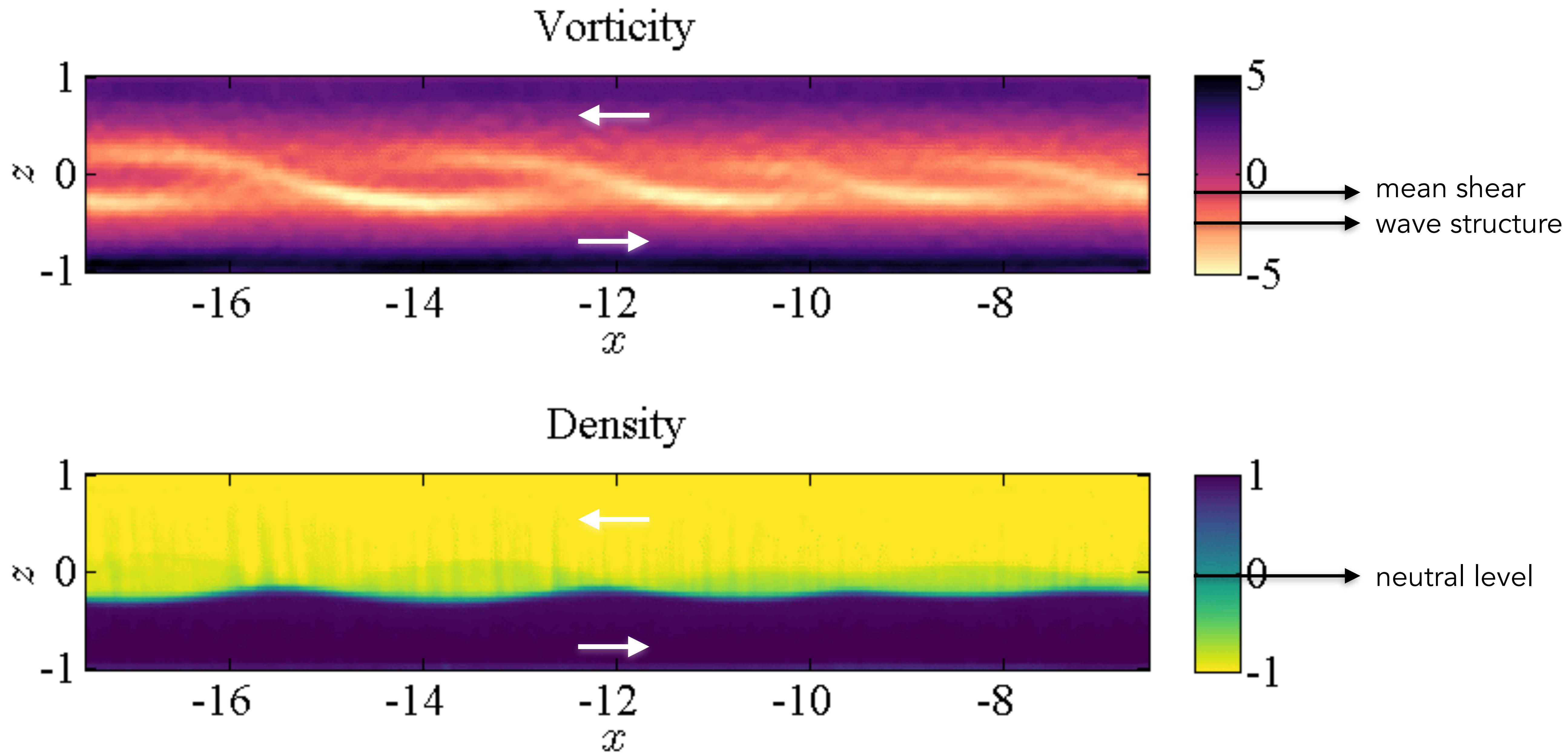

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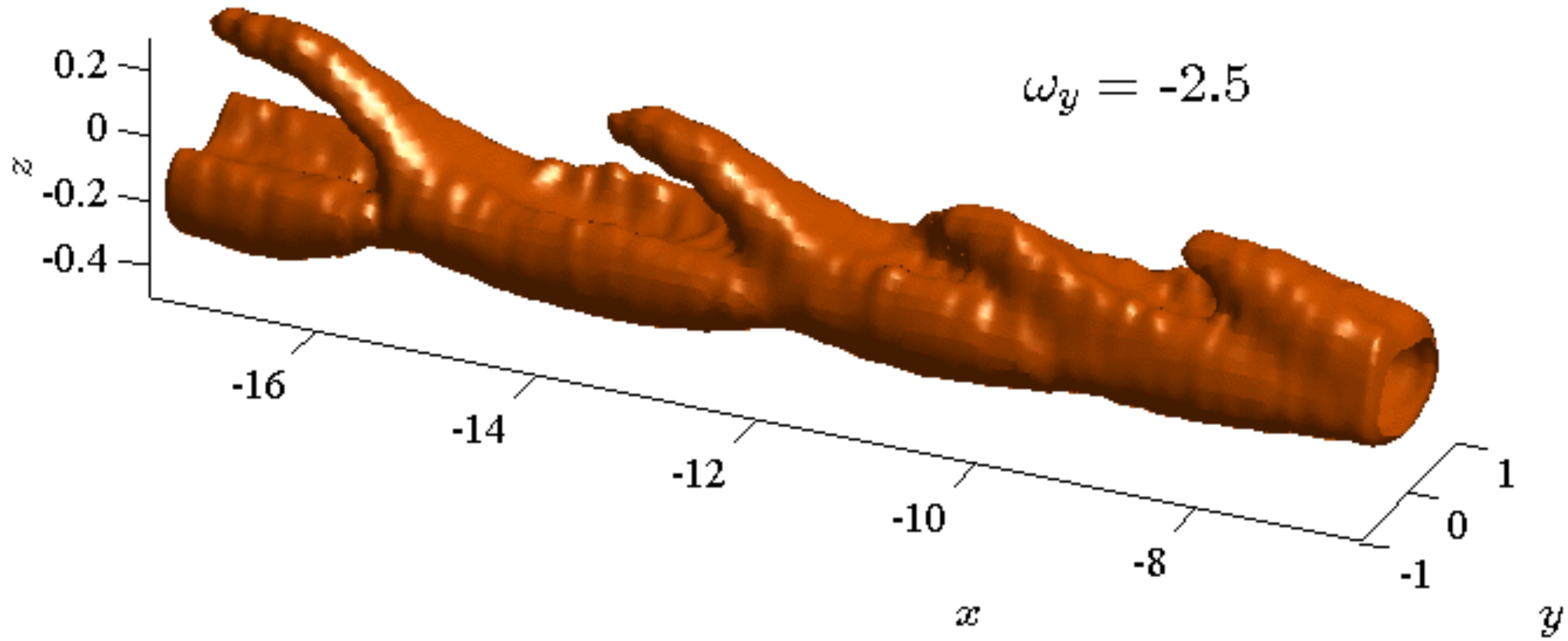

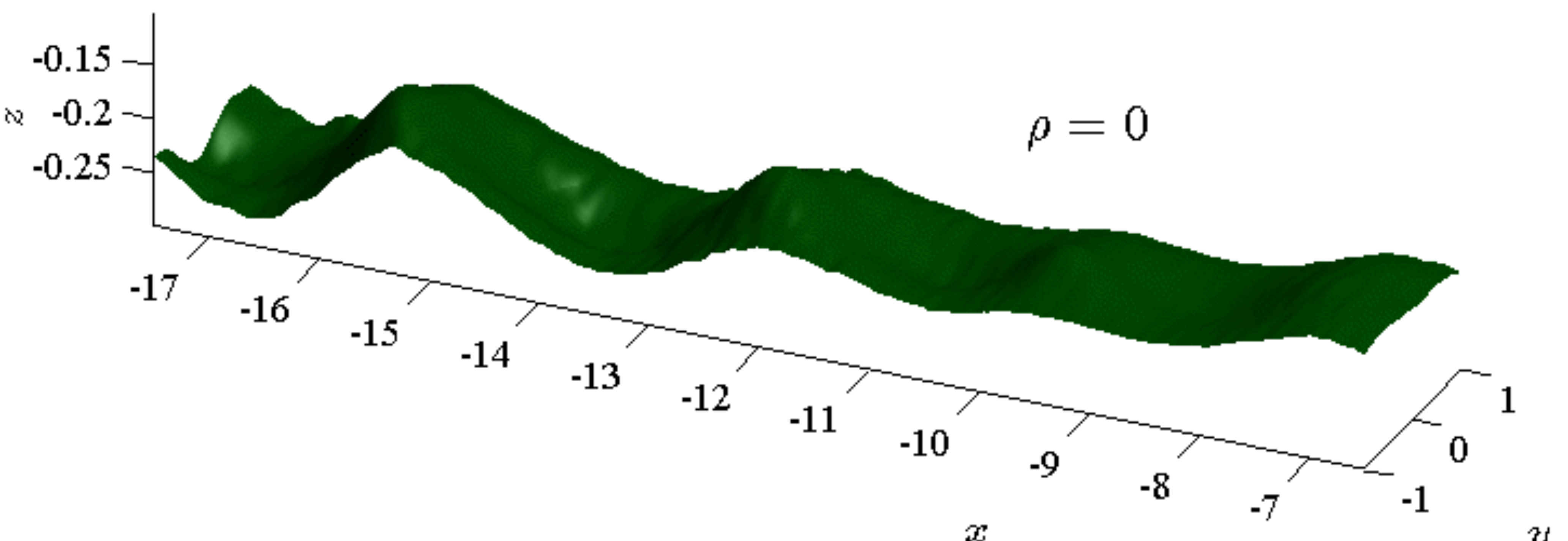

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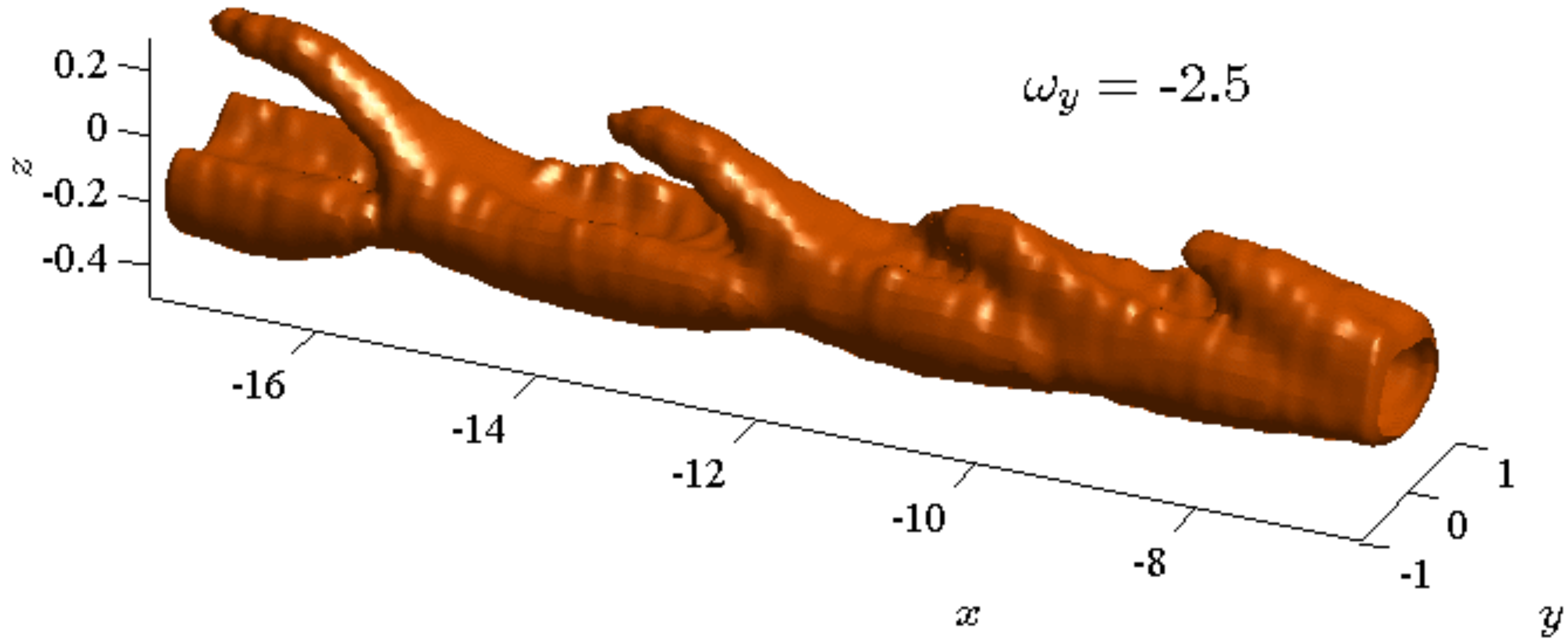

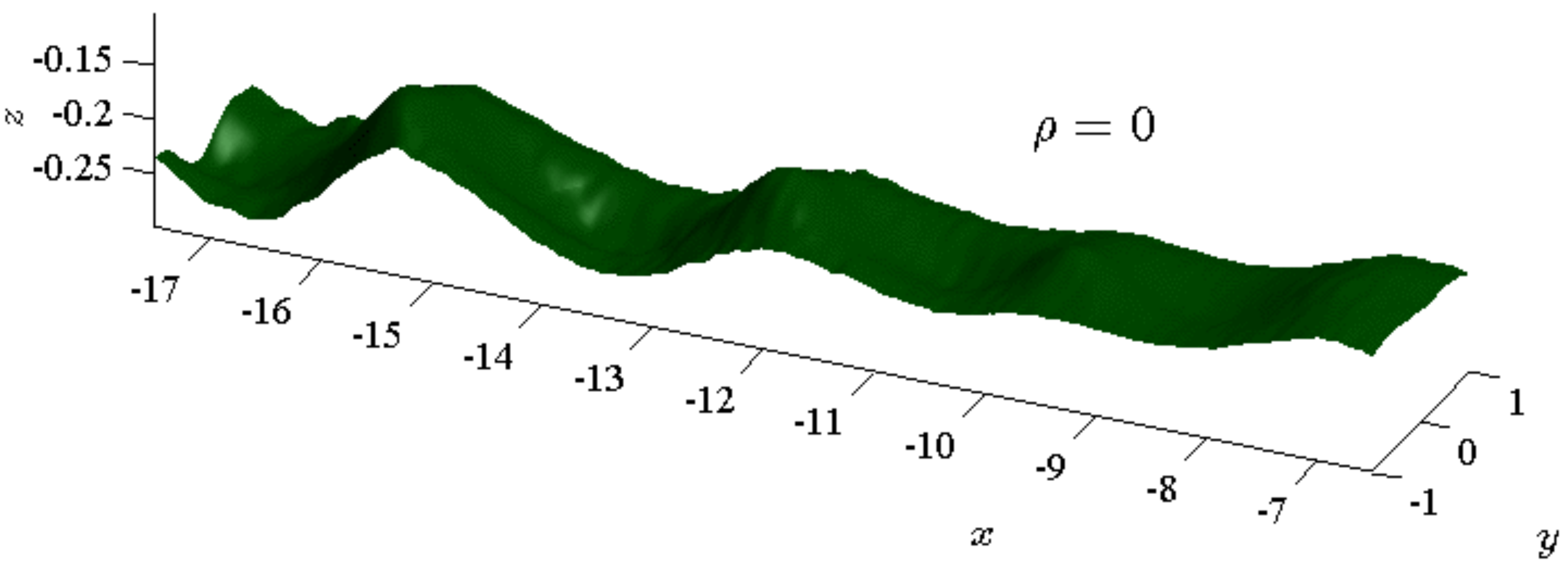

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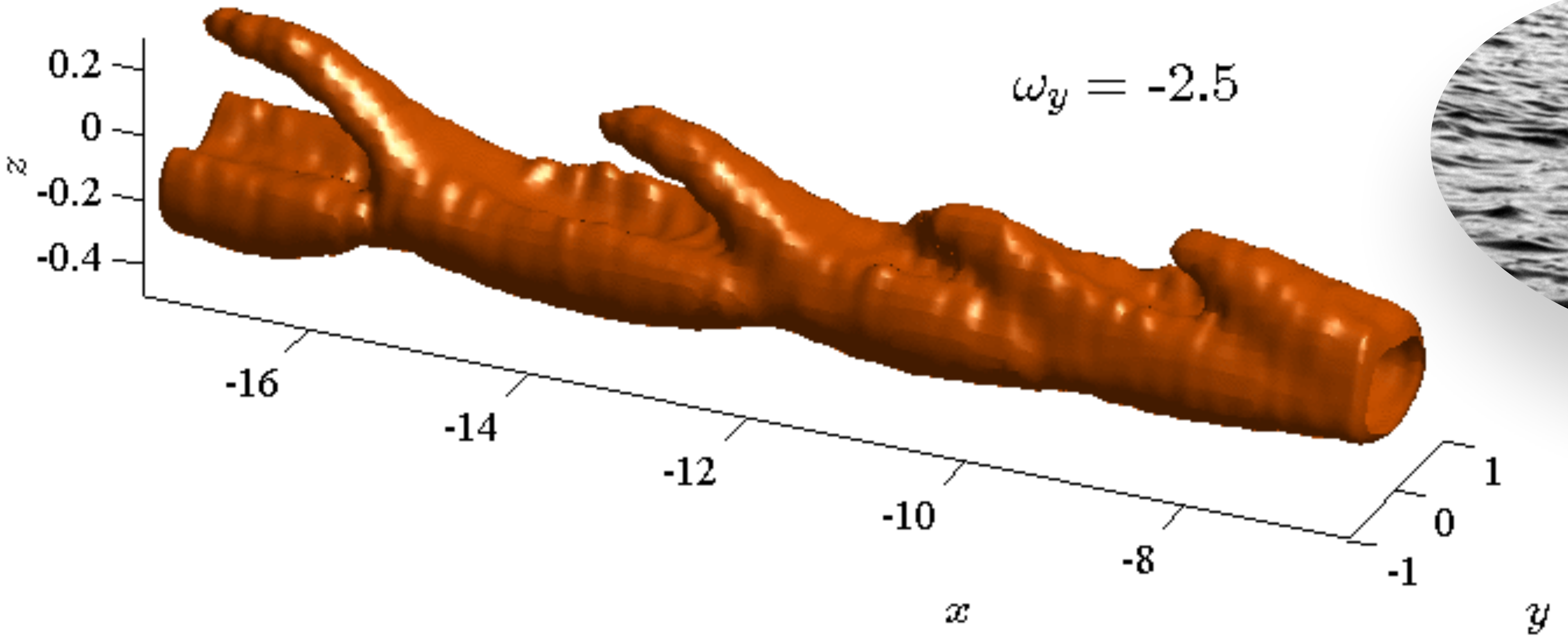
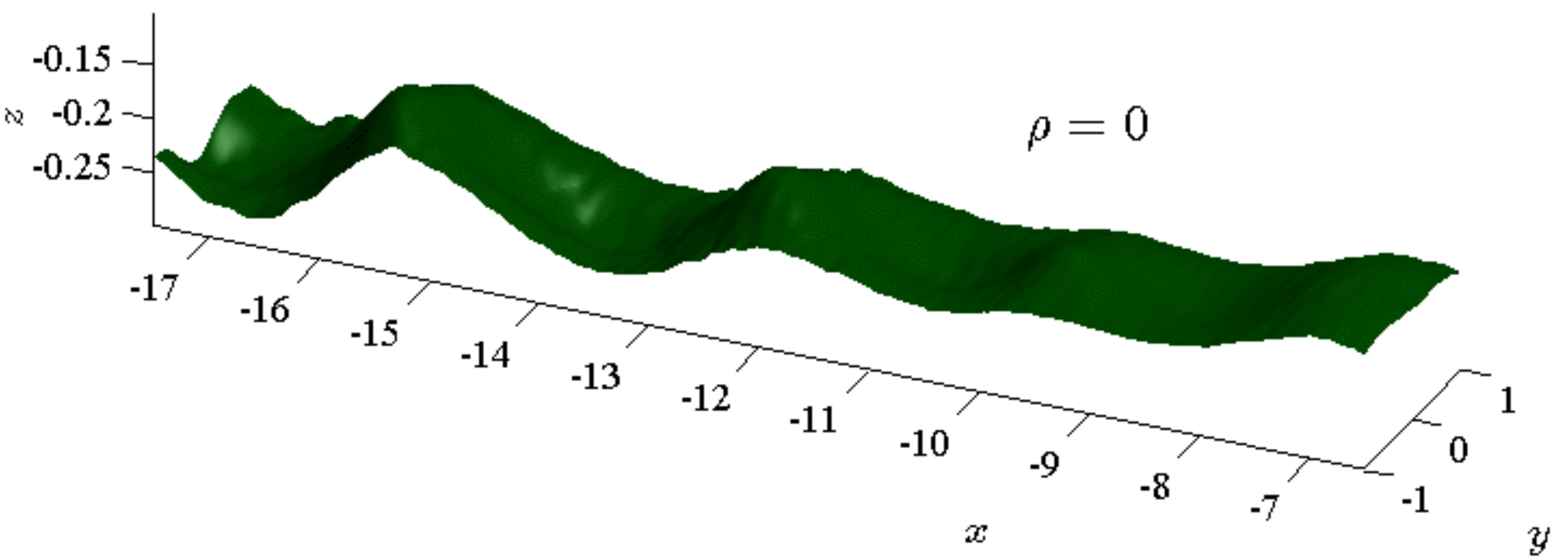


Nessie! ™ (Stuart Dalziel)

The Loch Ness monster



Mechanism: the Holmboe instability

Baines & Mitsudera (1994), Caulfield (1994)
Carpenter et al. (2013)

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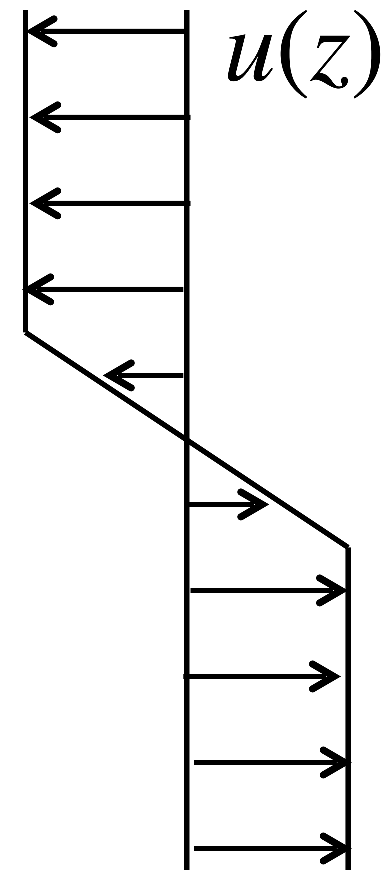

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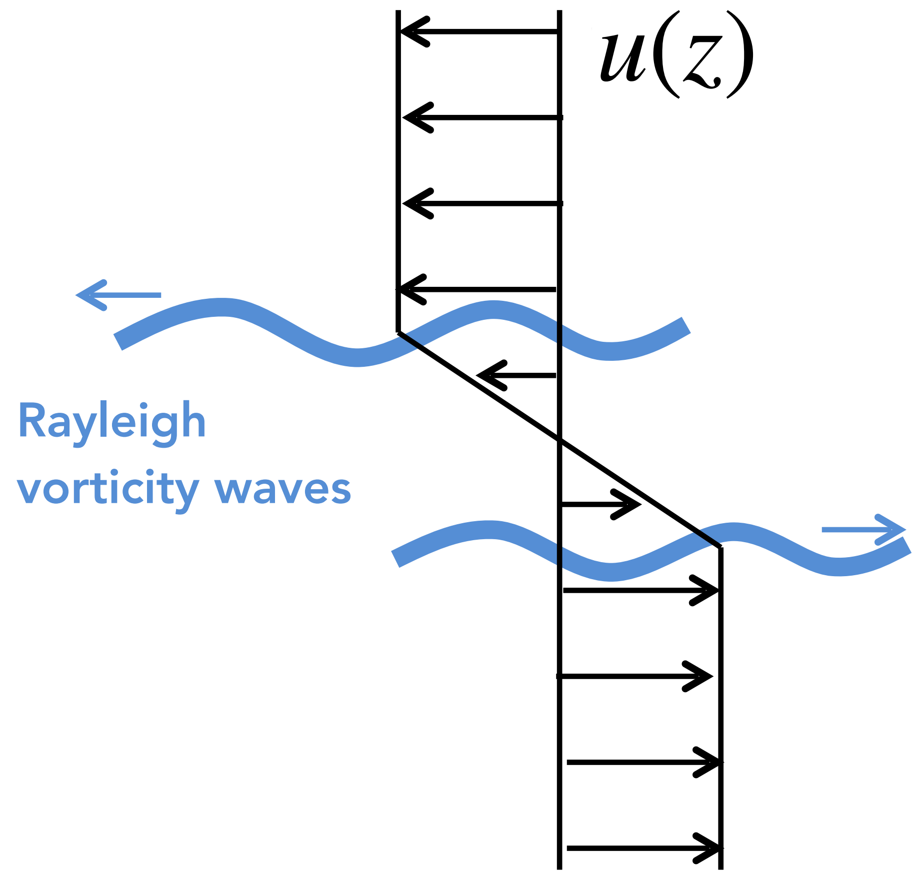

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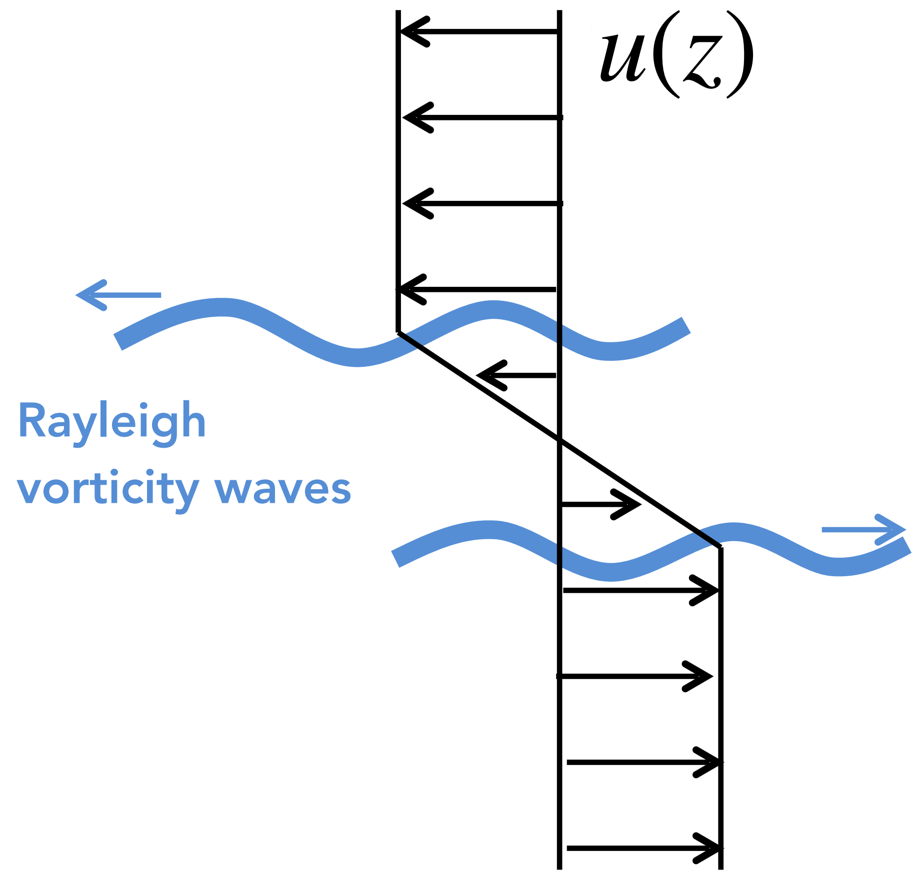

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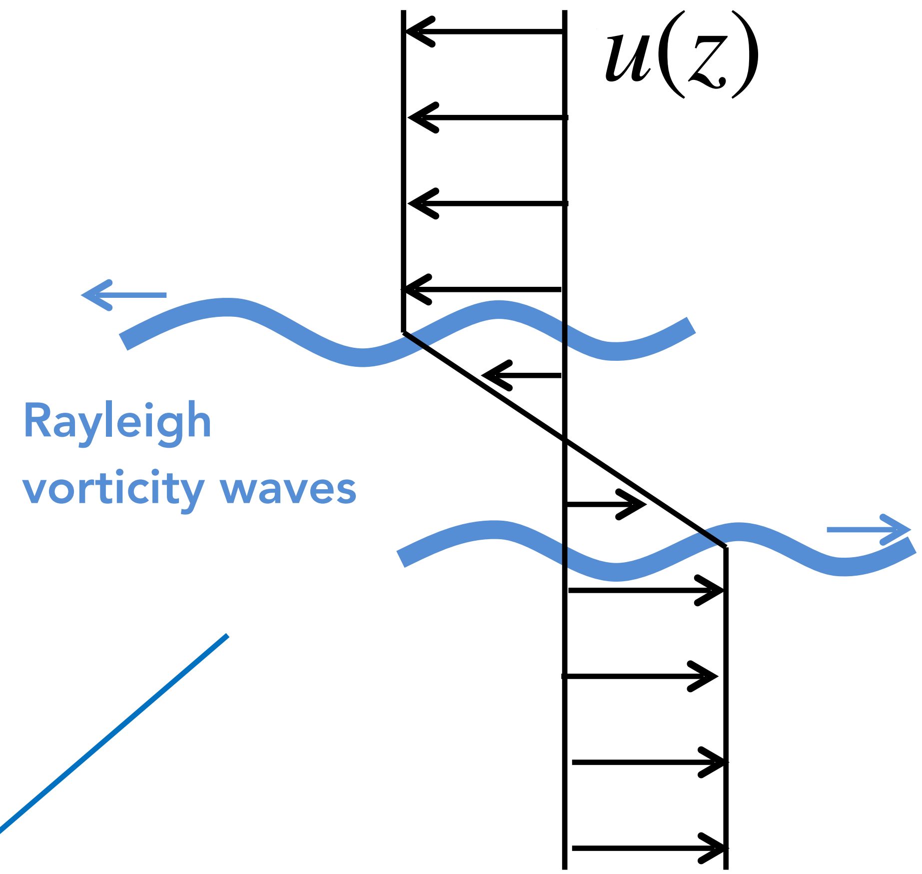
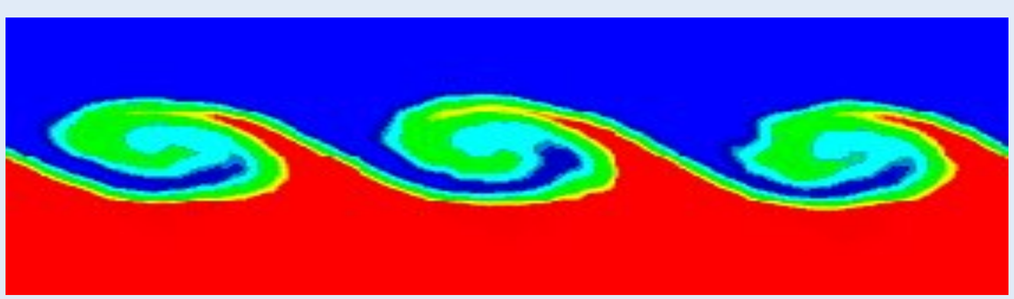


Kelvin-Helmholtz instability
(stationary)



suppressed by buoyancy forces

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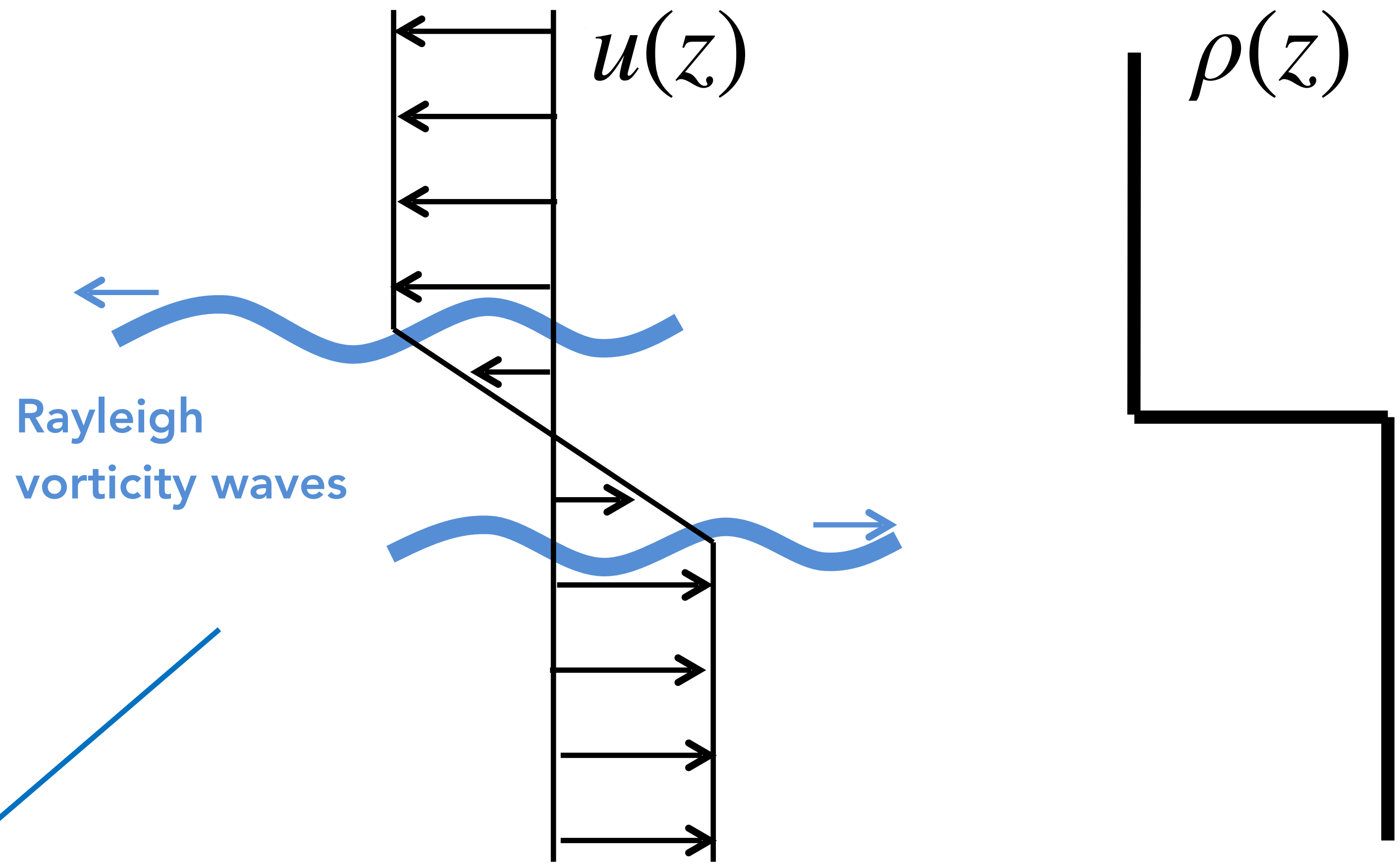

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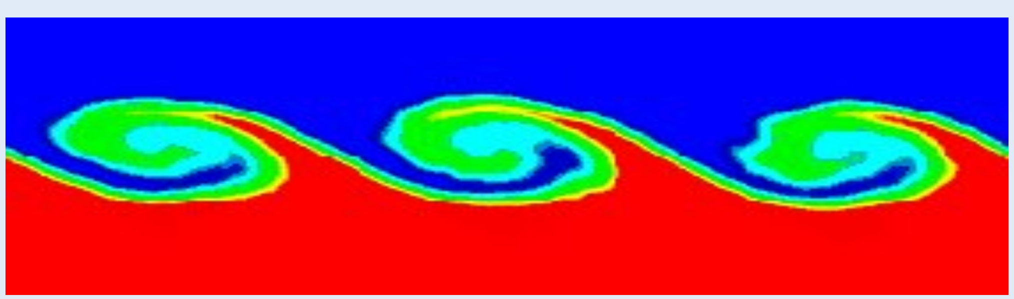

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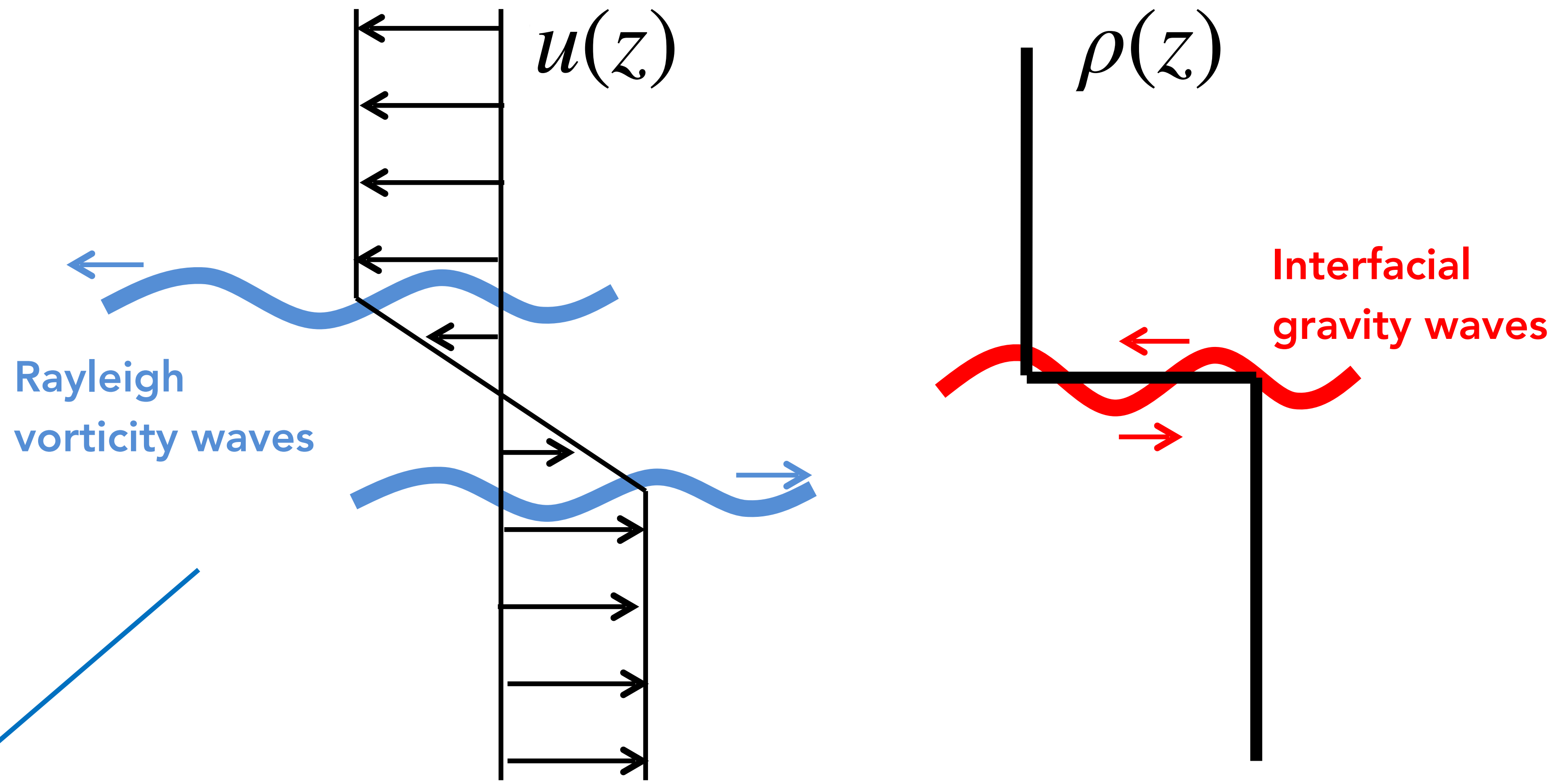
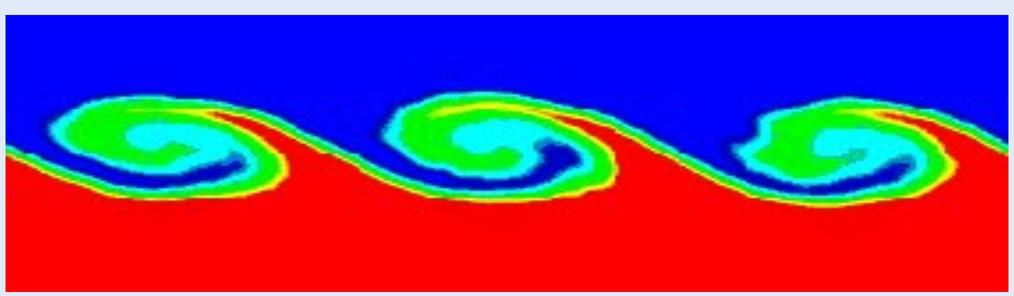


Kelvin-Helmholtz instability (stationary)



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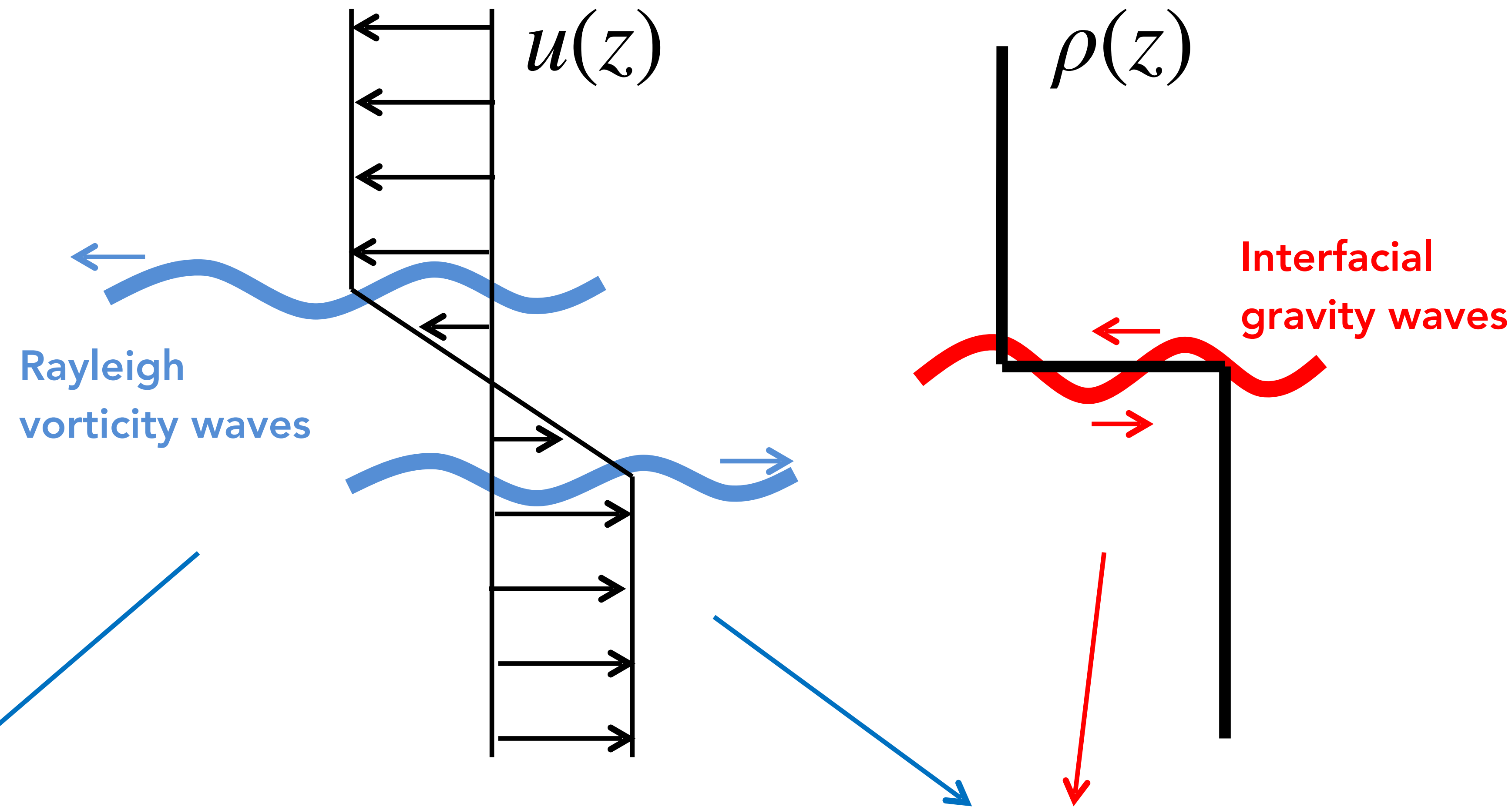

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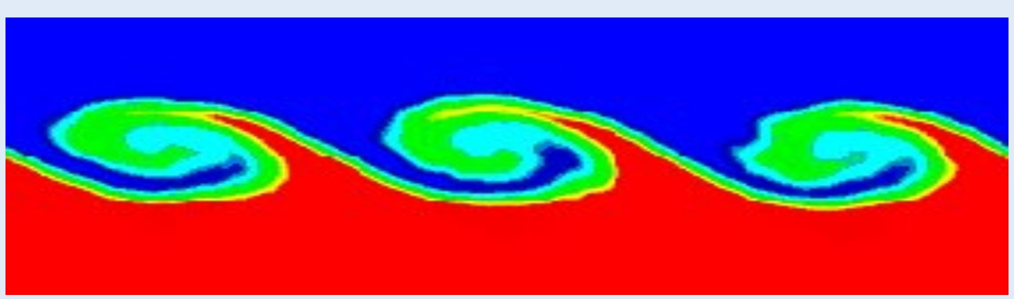
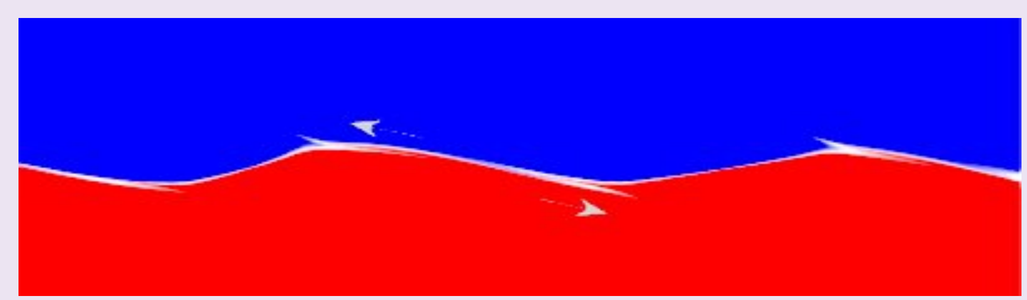


suppressed by buoyancy forces

Holmboe instability
(travelling)



enhanced by buoyancy forces

Mechanism: the Holmboe instability

Holmboe (1962)

- Classical analysis:

1. Assume: 1D base flow

$$\mathbf{u} = U(z)$$

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$$\mathbf{u} = U(z) \quad + \quad \varepsilon \begin{bmatrix} \hat{u}(z) \\ \hat{w}(z) \end{bmatrix} \exp(ikx + \sigma t)$$

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2. Linearise the Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{4}(-\cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\mathbf{x}})\rho + \frac{1}{Re}\nabla^2 \mathbf{u}$$

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = \frac{1}{Re\,Pr}\nabla^2 \rho$$

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- Squire's/Yih's theorem: fastest growing mode is **2D**

  i.e. modes at an angle $\mathbf{k} = |k|(\cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{y}})$ have slower growth σ_r

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- This does not apply to **2D base flows** and **confinement by boundaries!**

3D linear stability

Assume $$\mathbf{u} = U(y,z) + \varepsilon \begin{bmatrix} \hat{u}(y,z) \\ \hat{v}(y,z) \\ \hat{w}(y,z) \end{bmatrix} \exp(ikx + \sigma t)$$

$(y,z) \in [-1,1] \times [-1,1]$

\+ no-slip condition at walls

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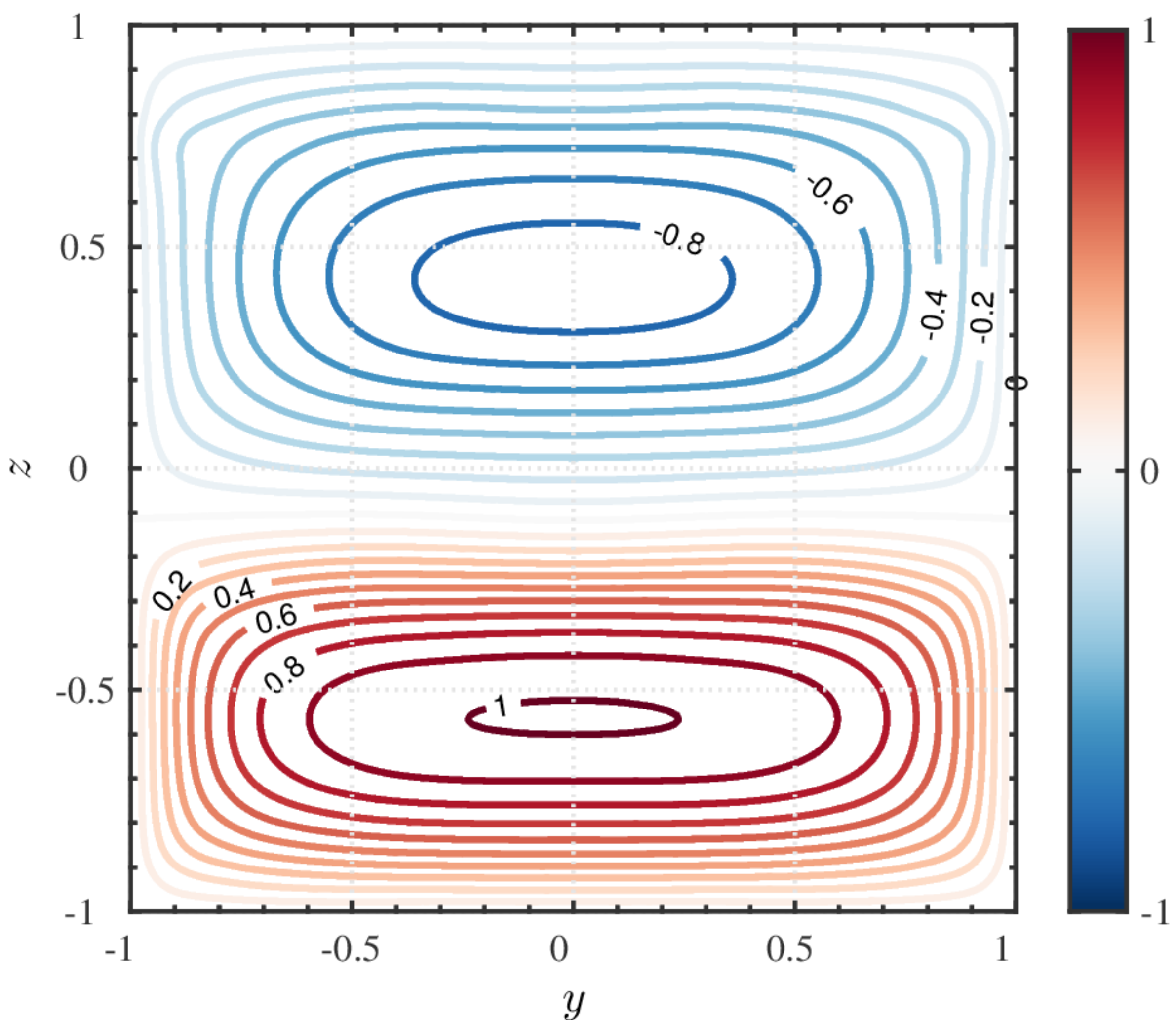
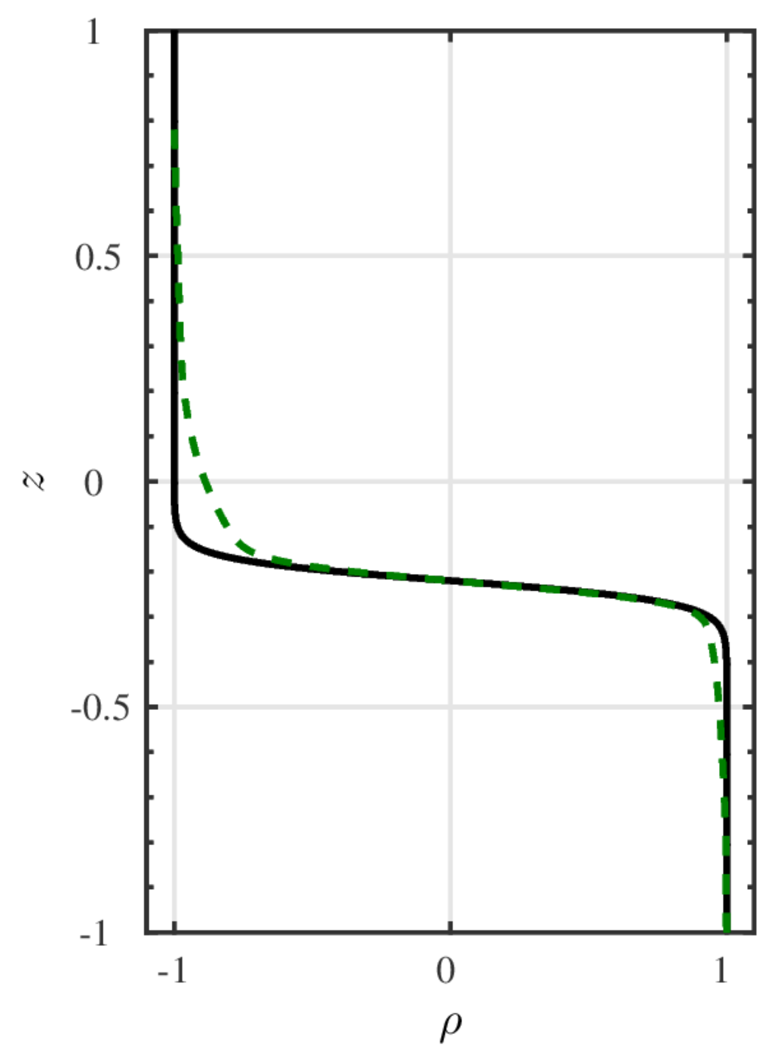
Experimental mean flows: $U(y,z)$ $\qquad$ $R(z)$

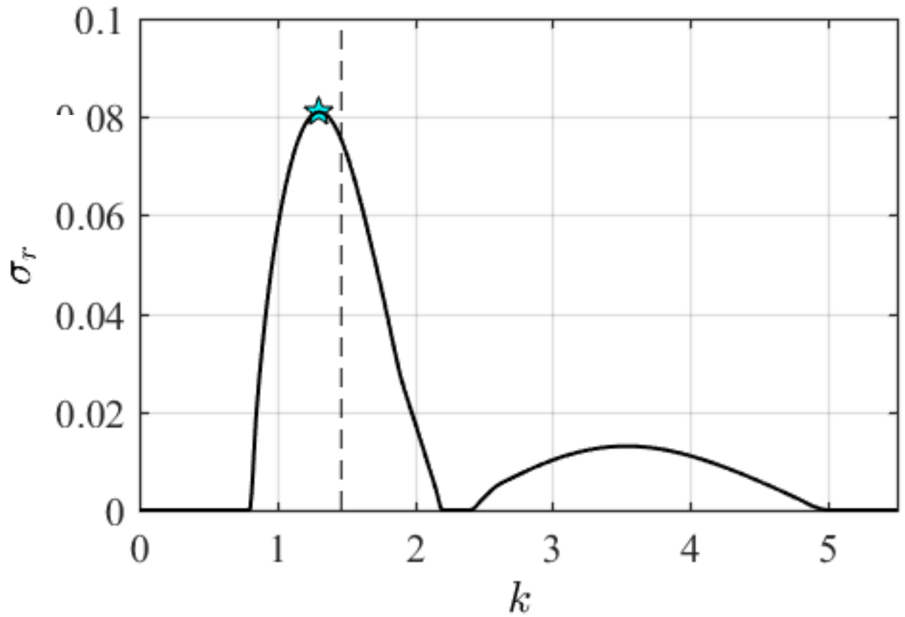




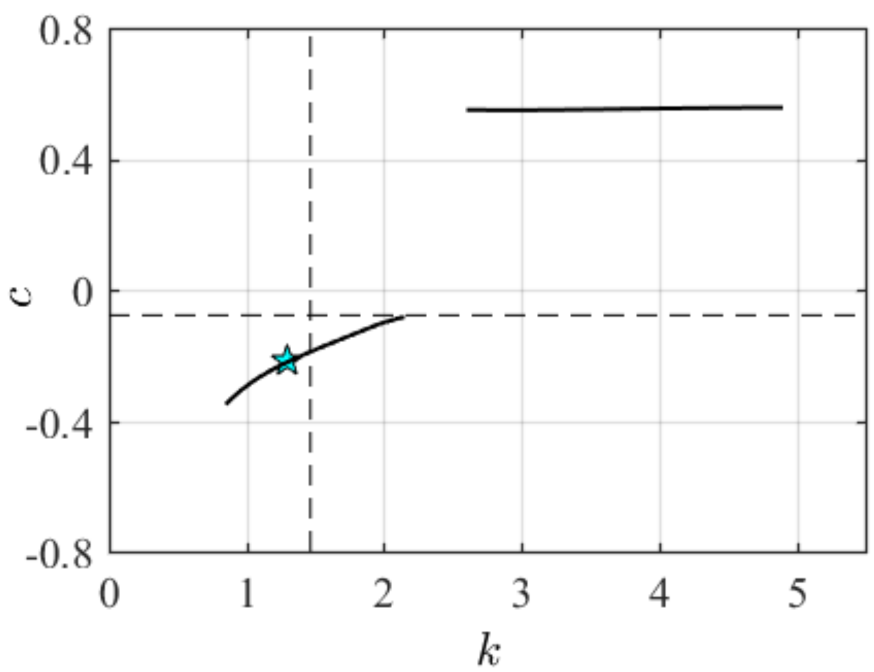
3D linear stability: theory vs experiment

Theory: unstable Holmboe mode

Growth rate σ_r



Phase speed c



$k = 1.32$

$c = -0.21$

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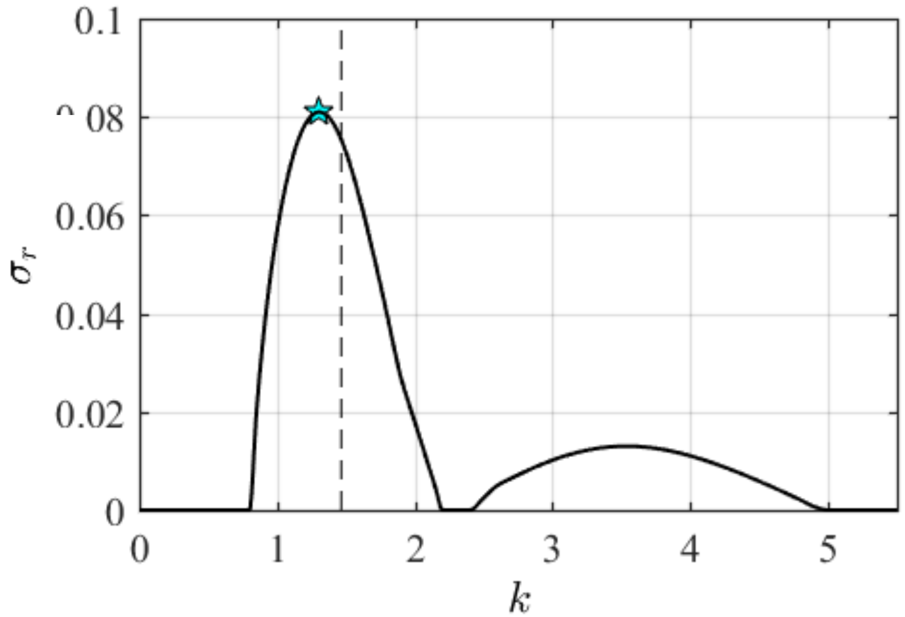

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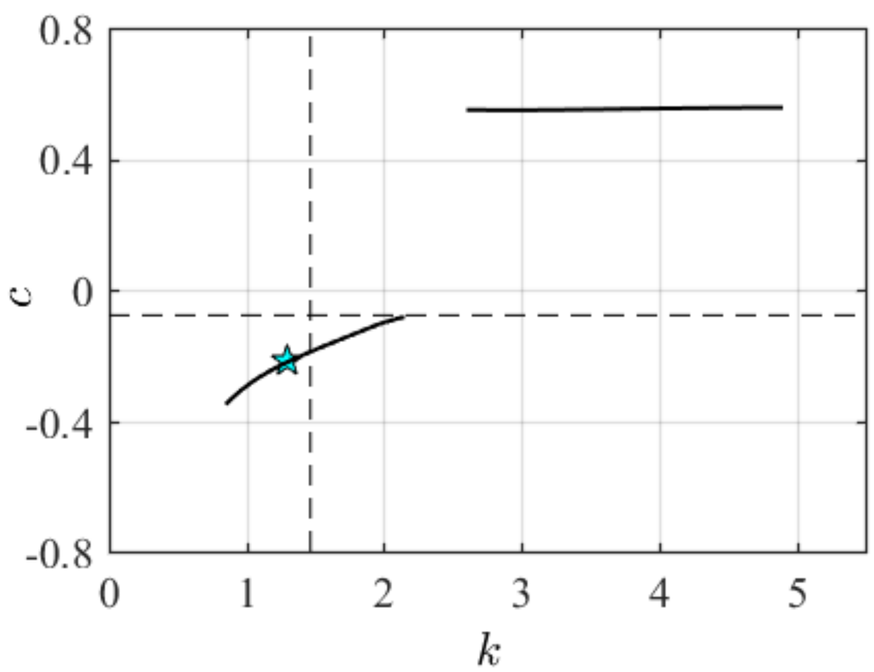

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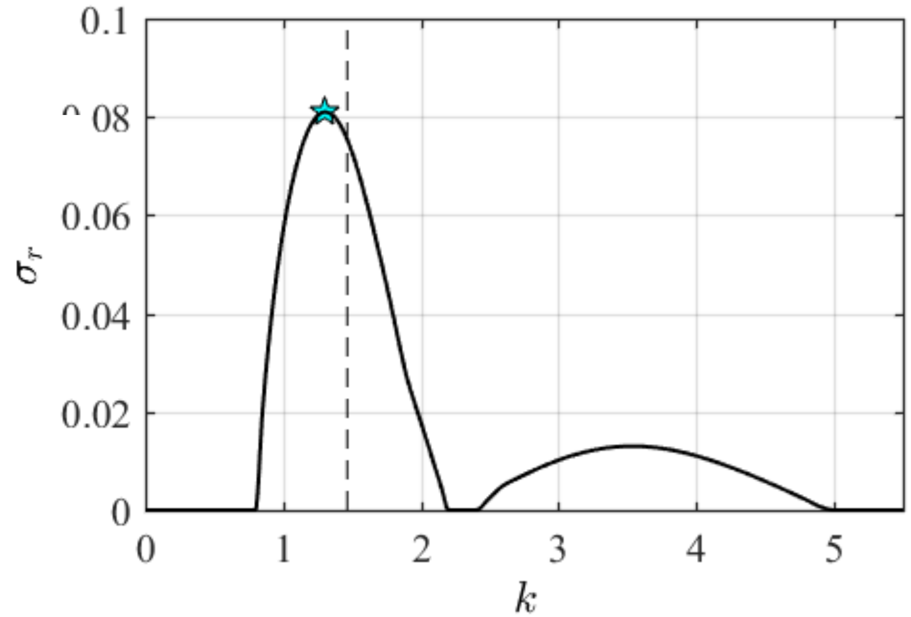

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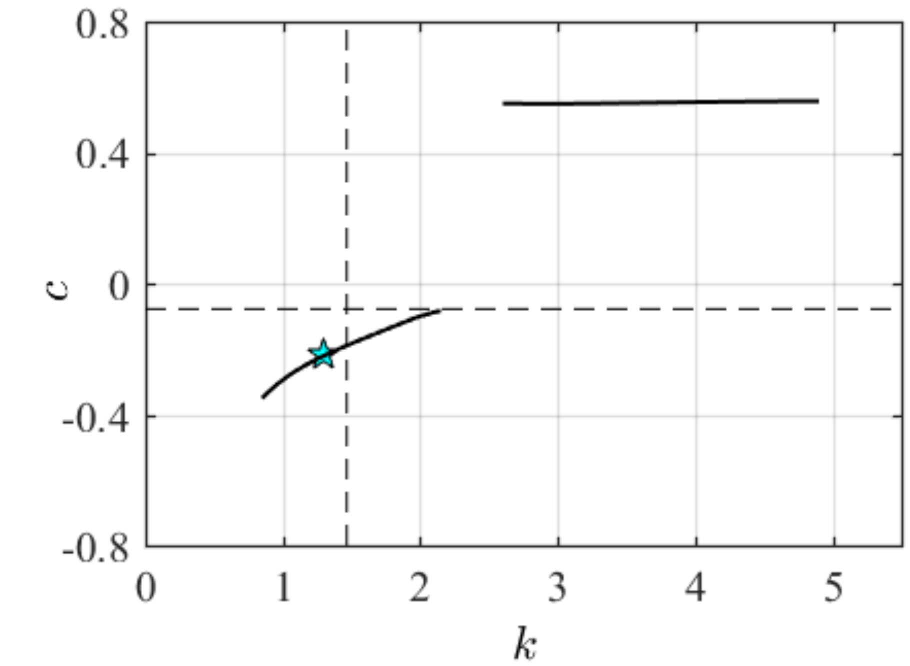

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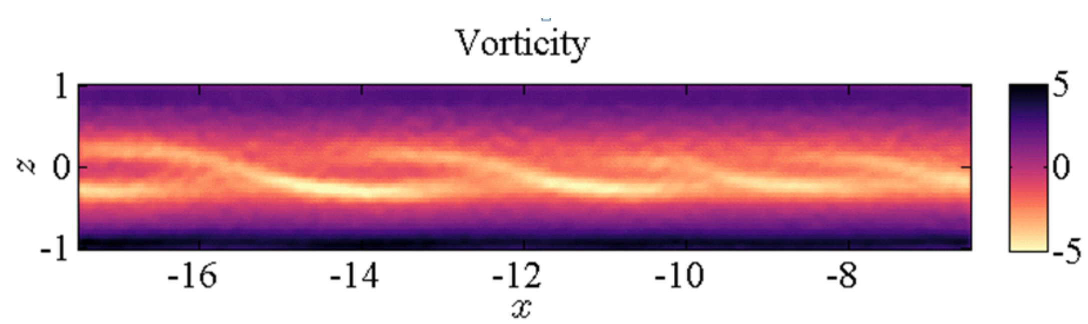

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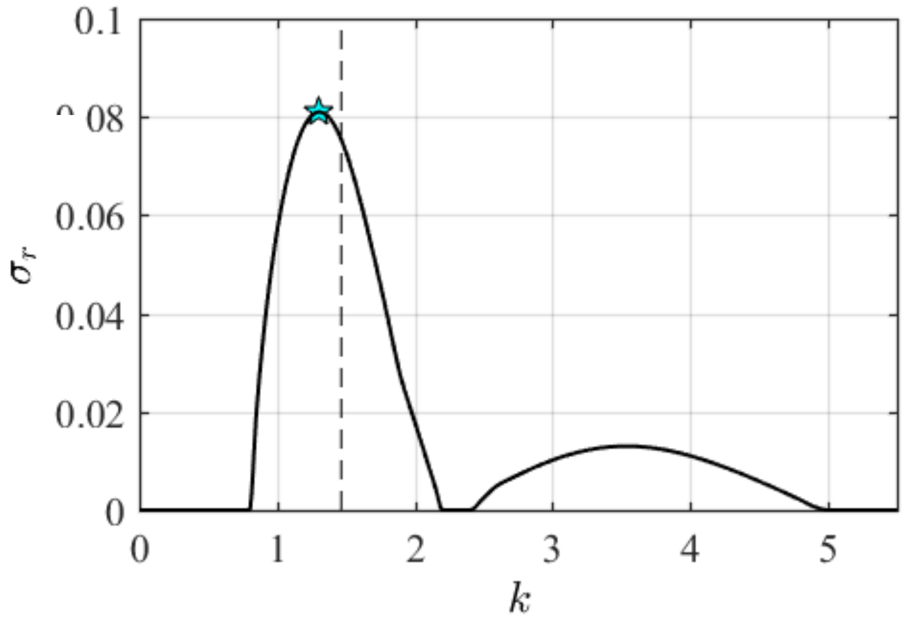

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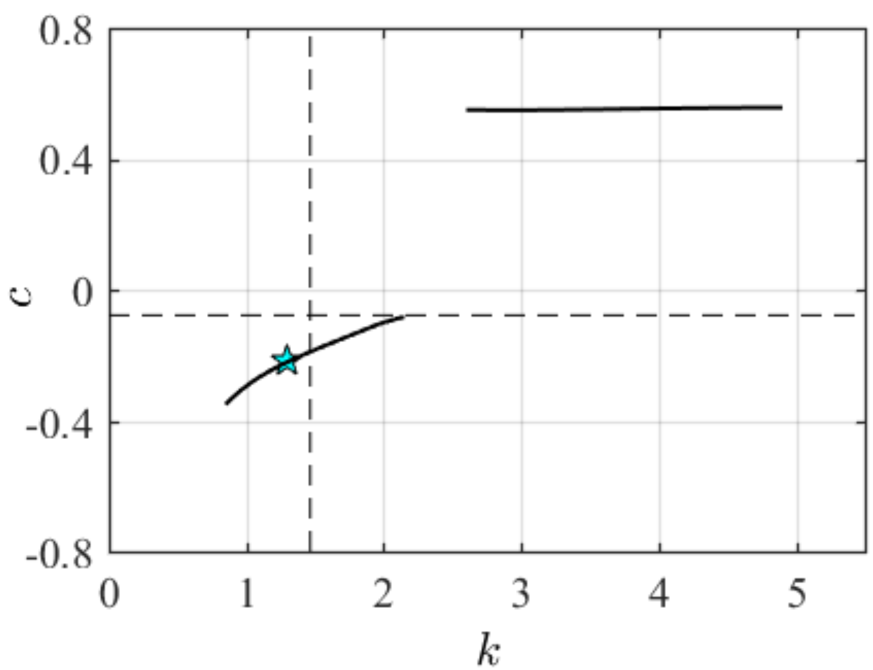

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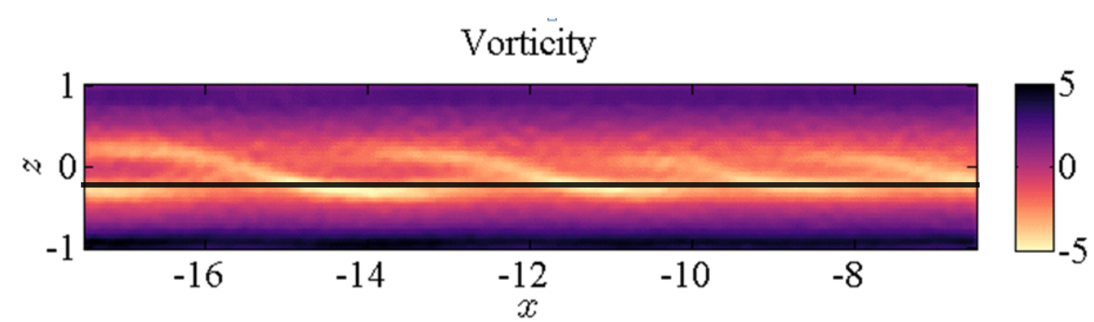

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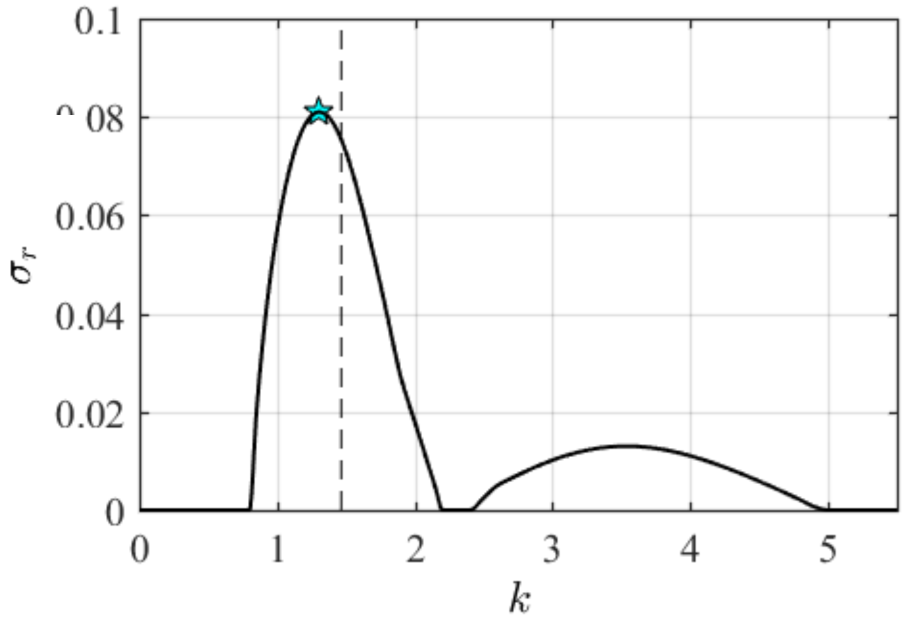

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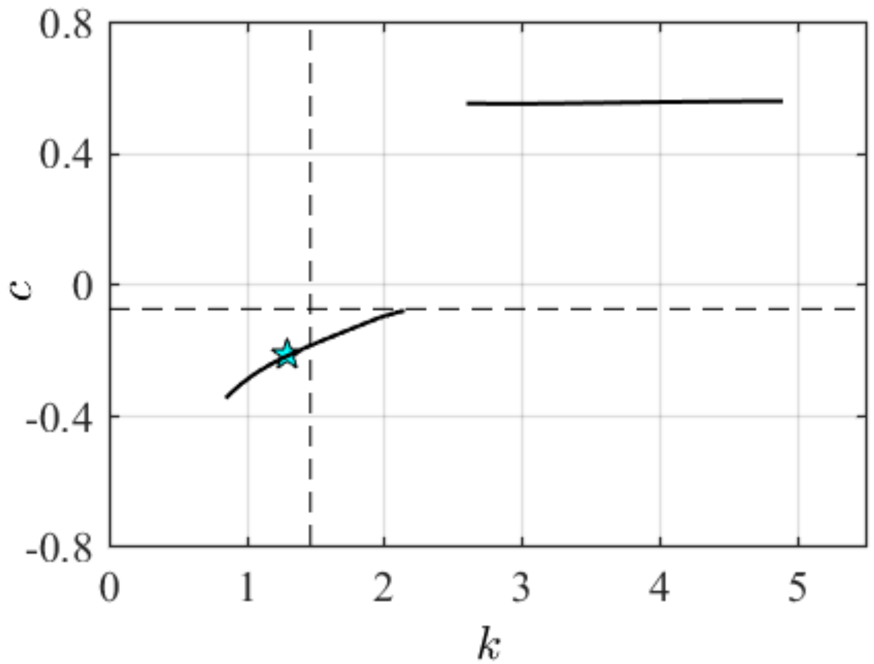

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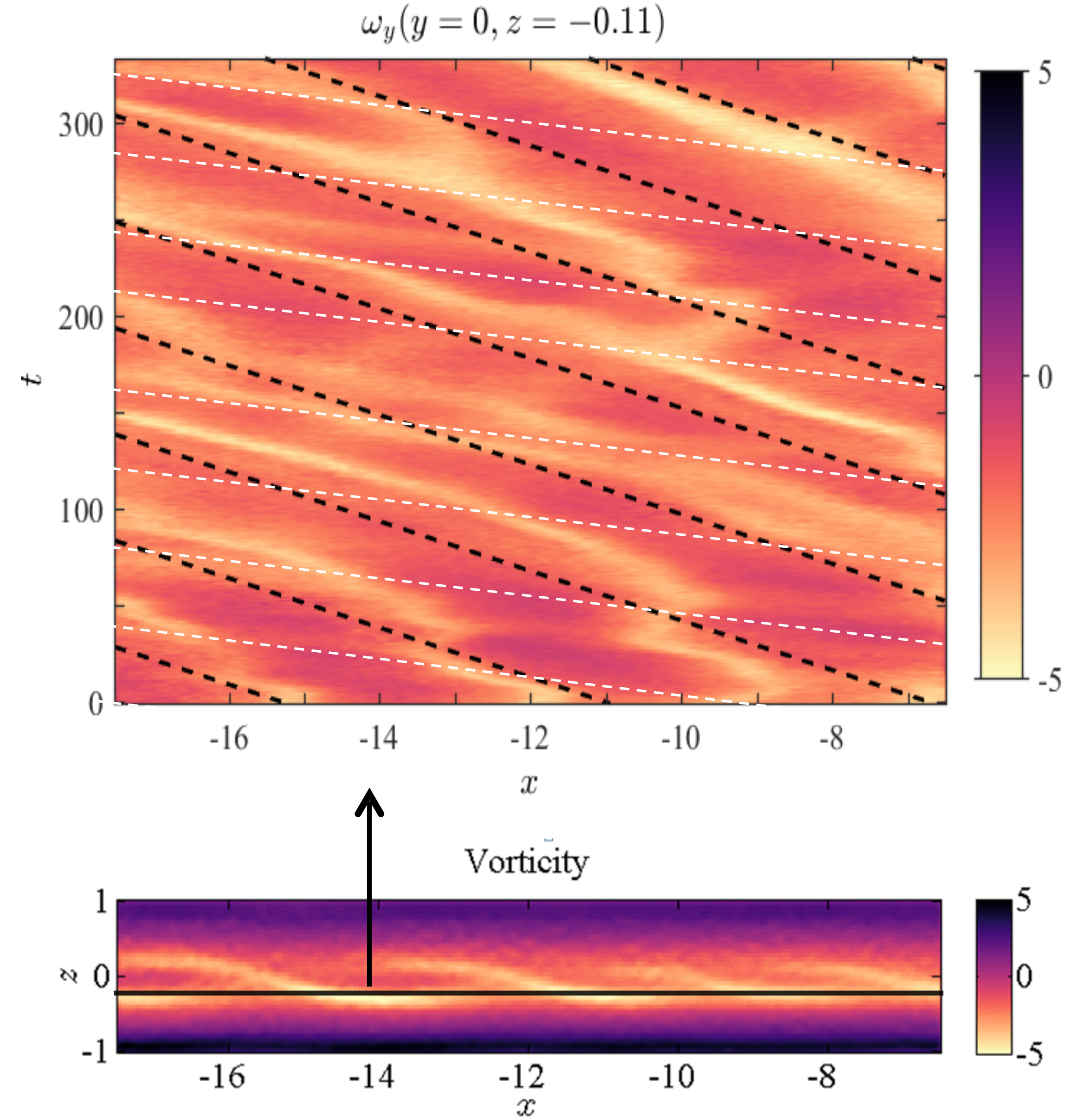

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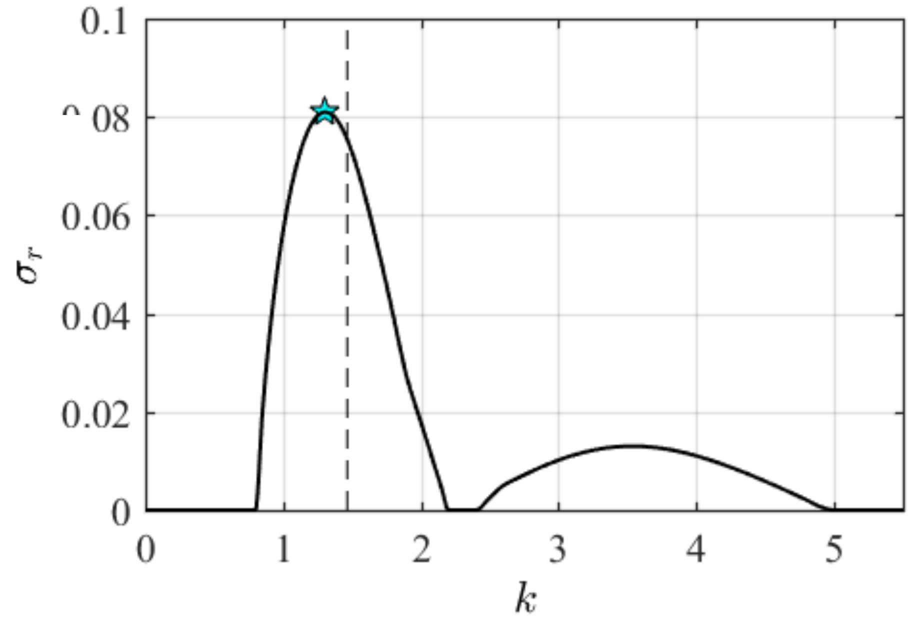

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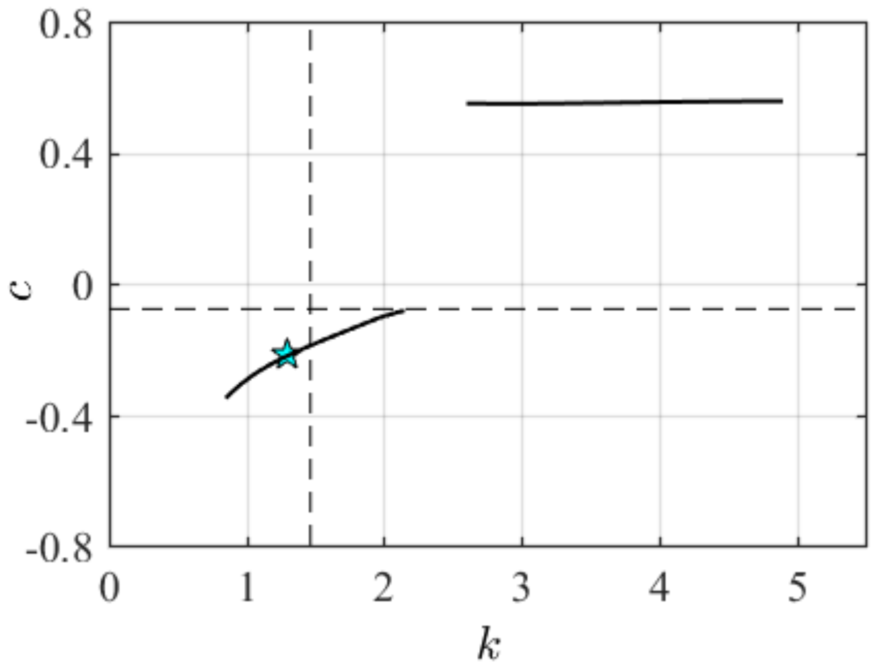

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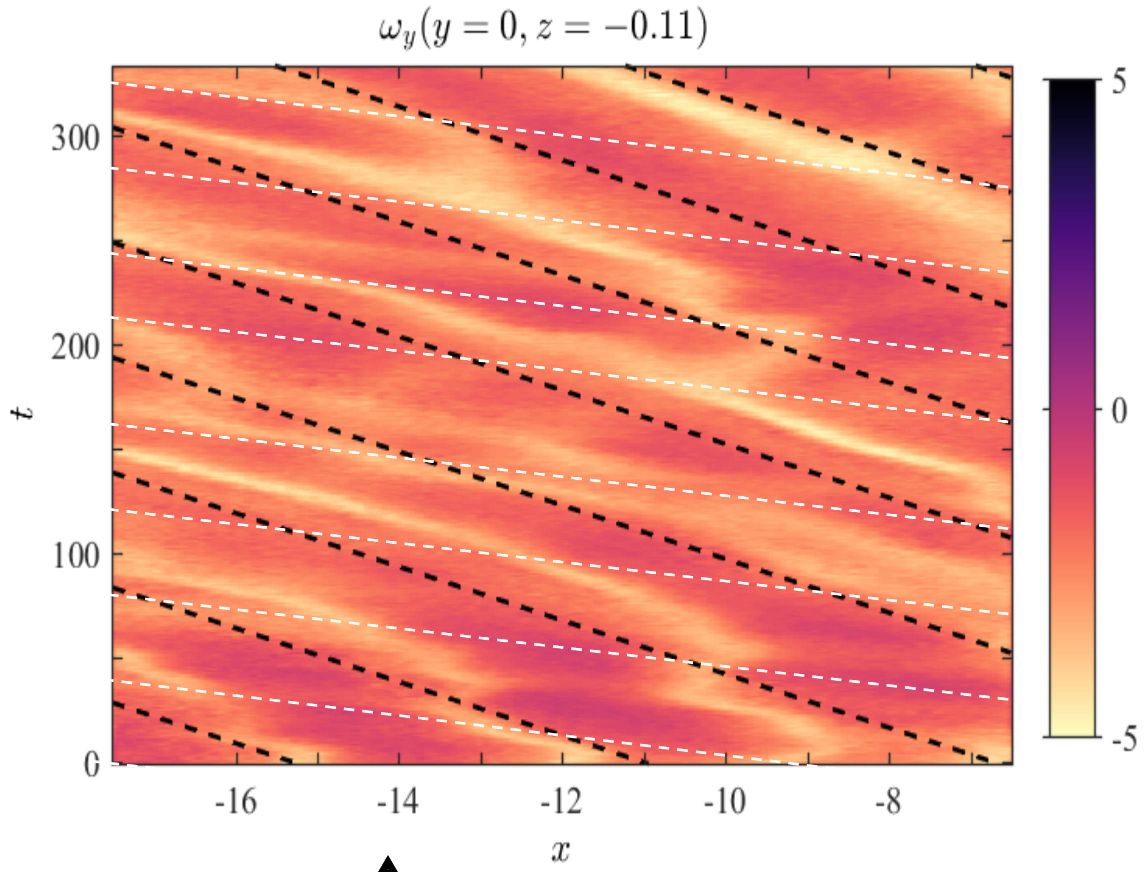

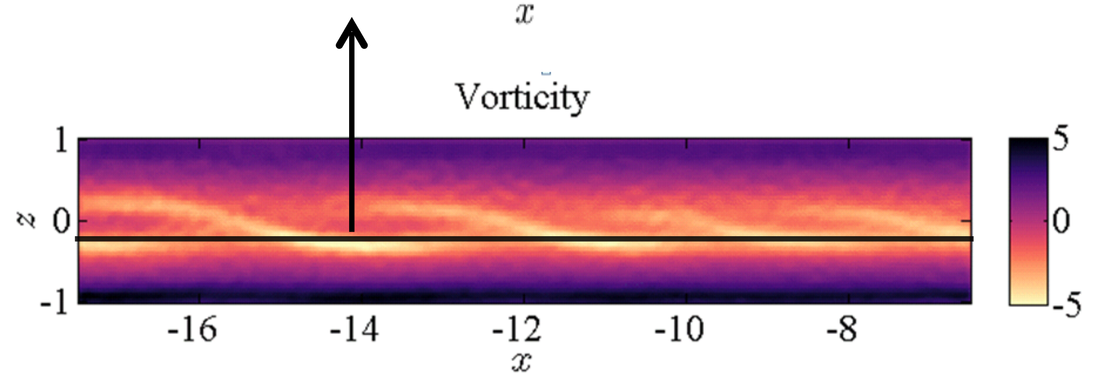

vs

$k \in [1.05, 1.80]$

$c \in [-0.20, -0.08]$

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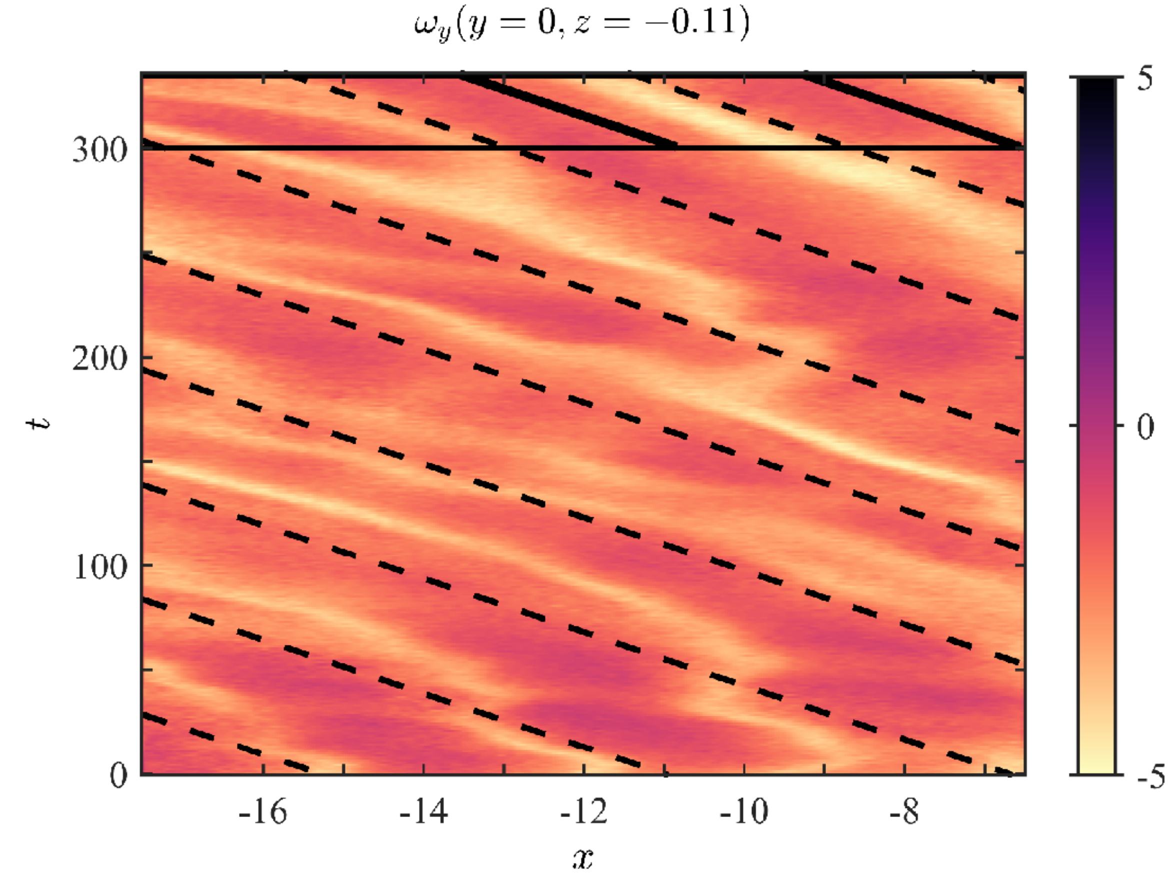

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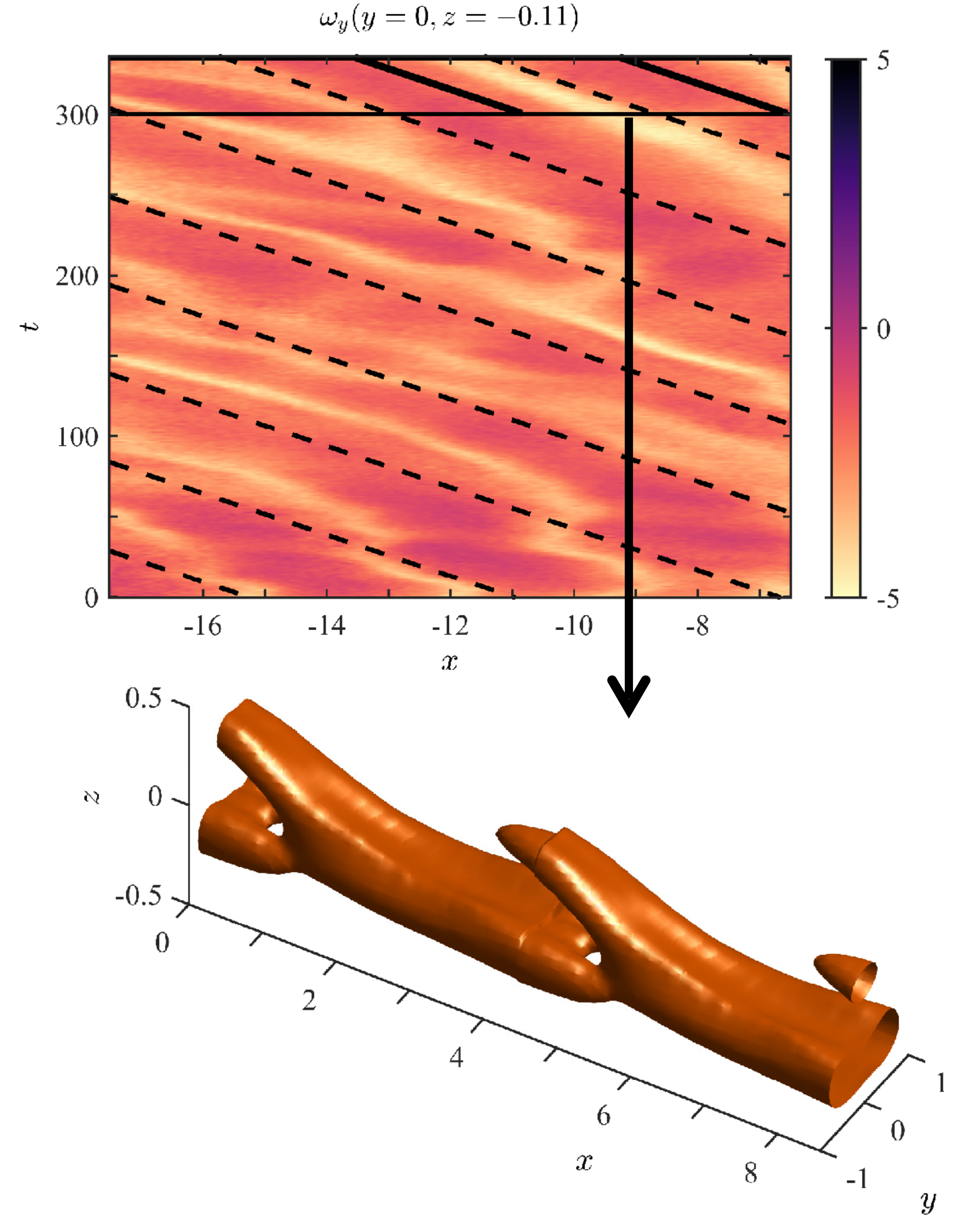

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$$0 < \varepsilon \ll 1$$

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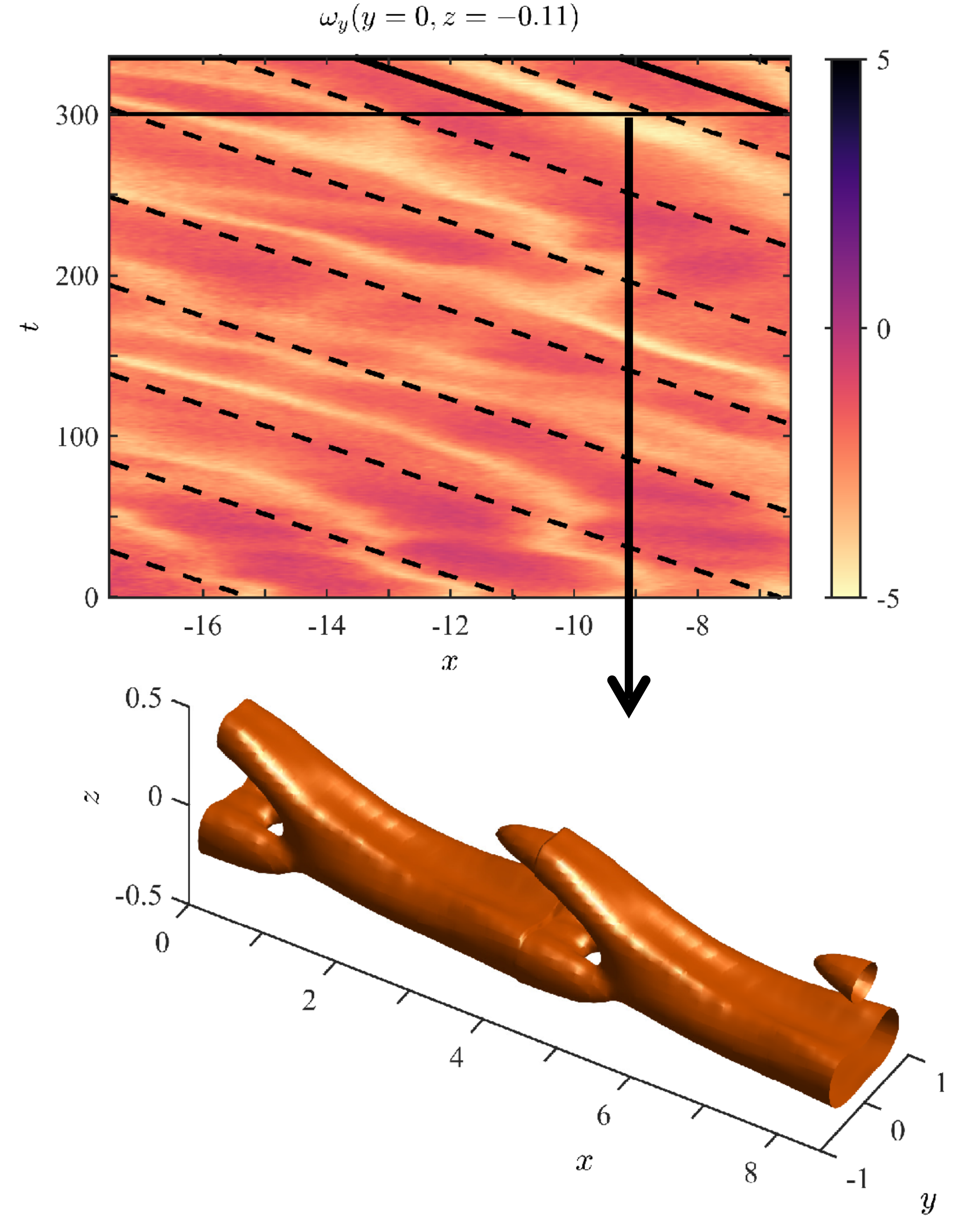

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$$\mathbf{u} = U(y,z) + \alpha \begin{bmatrix} \hat{u}(y,z) \\ \hat{v}(y,z) \\ \hat{w}(y,z) \end{bmatrix} \exp(ikx + \phi)$$

$\alpha = O(0.1)$

determined to match $\langle \omega_y \rangle_{x,y,z}^{\text{rms}}$ with experiment

'confined Holmboe instability'

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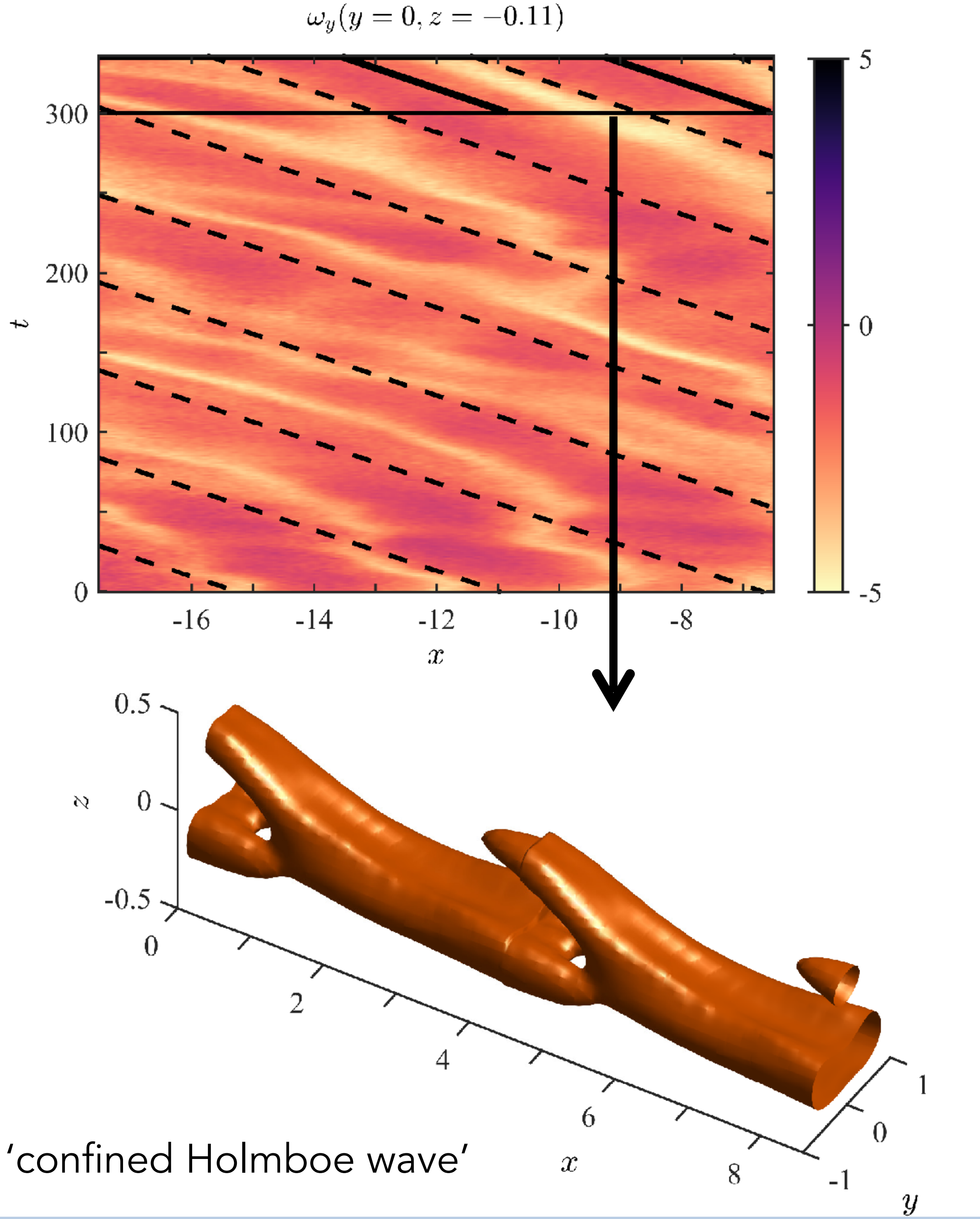

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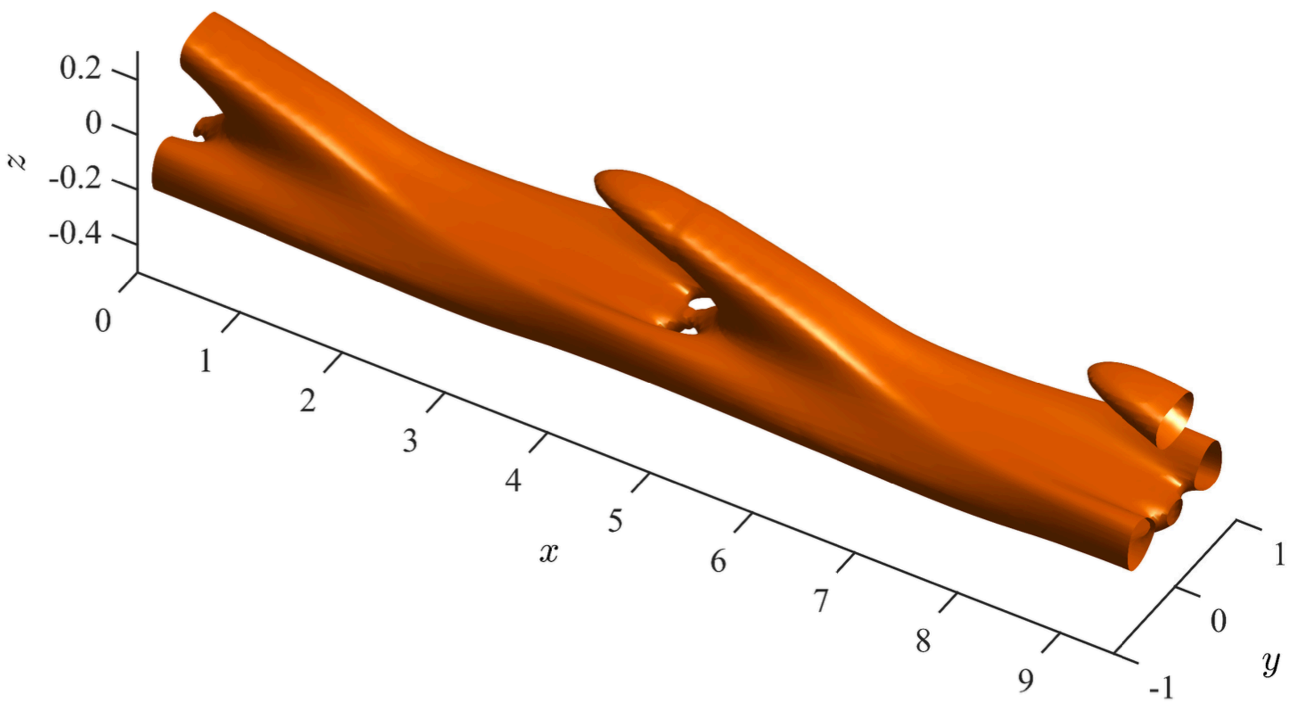

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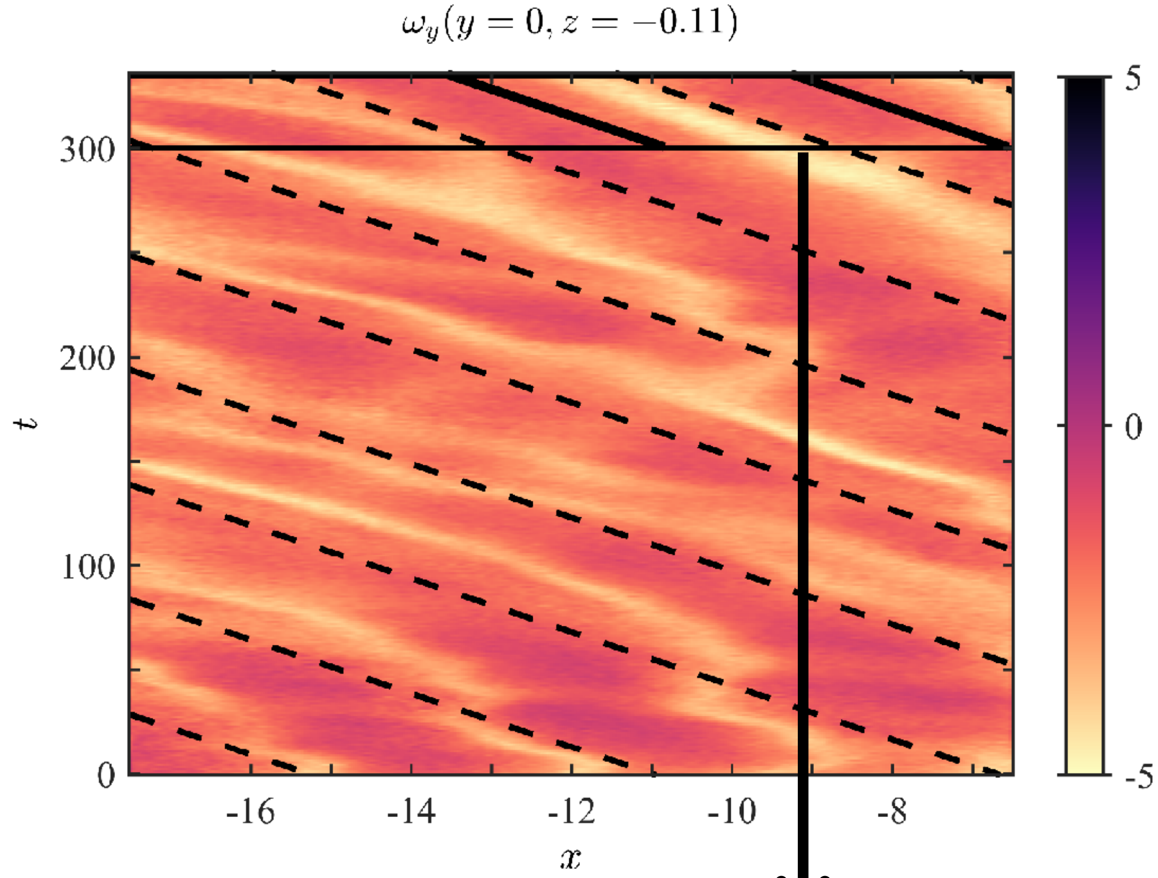

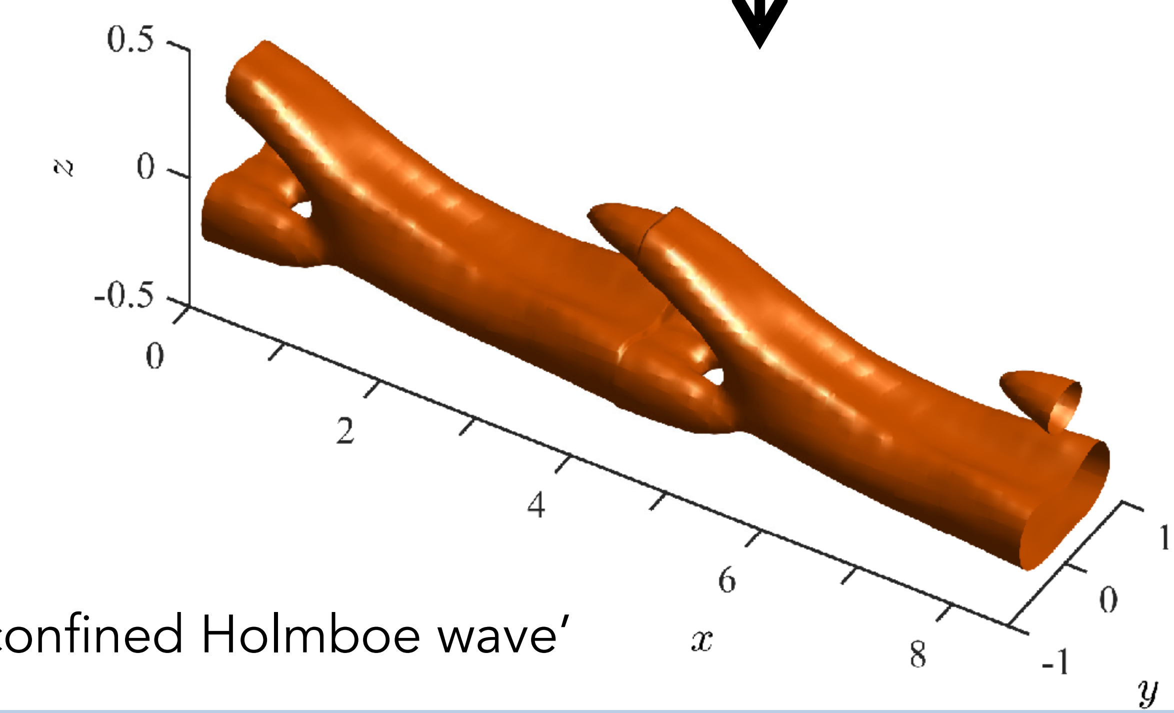

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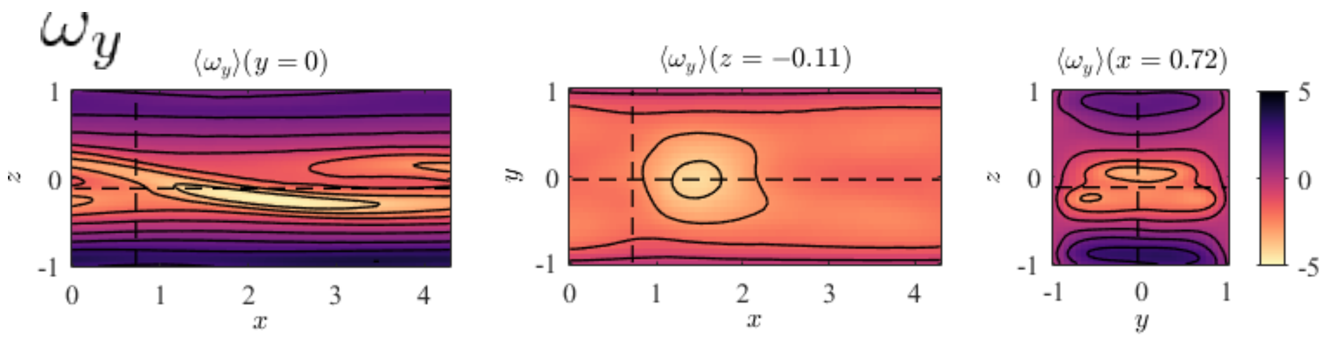

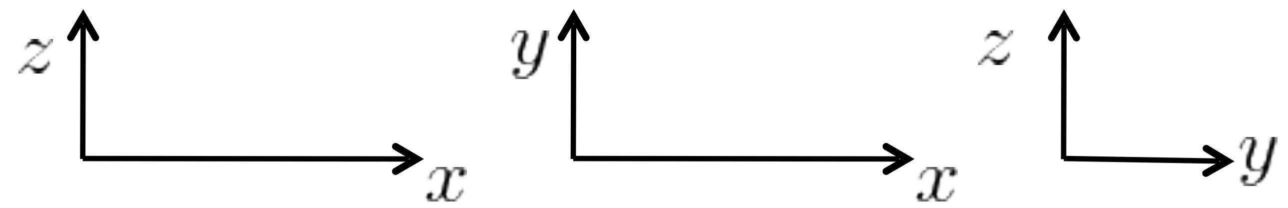

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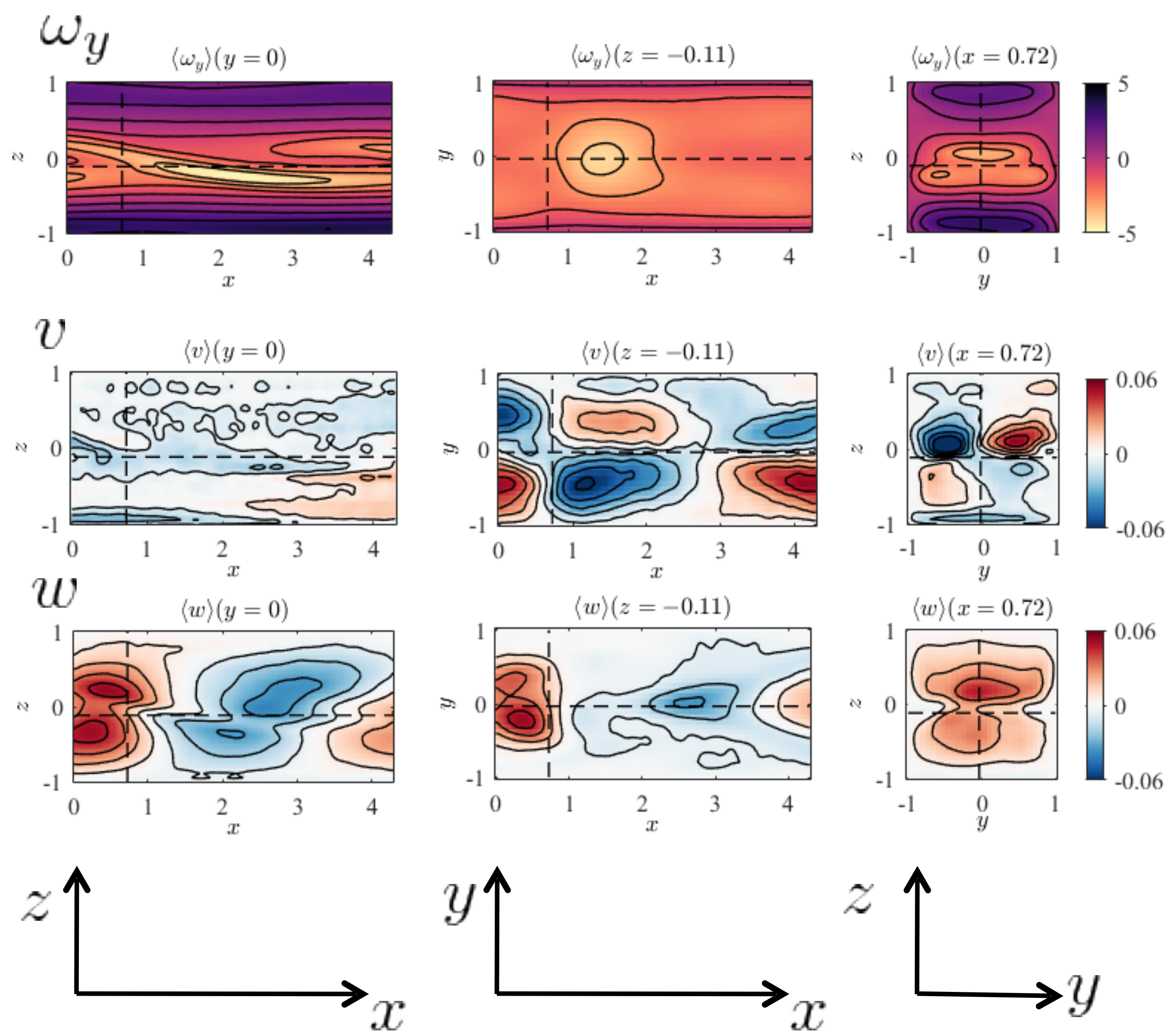

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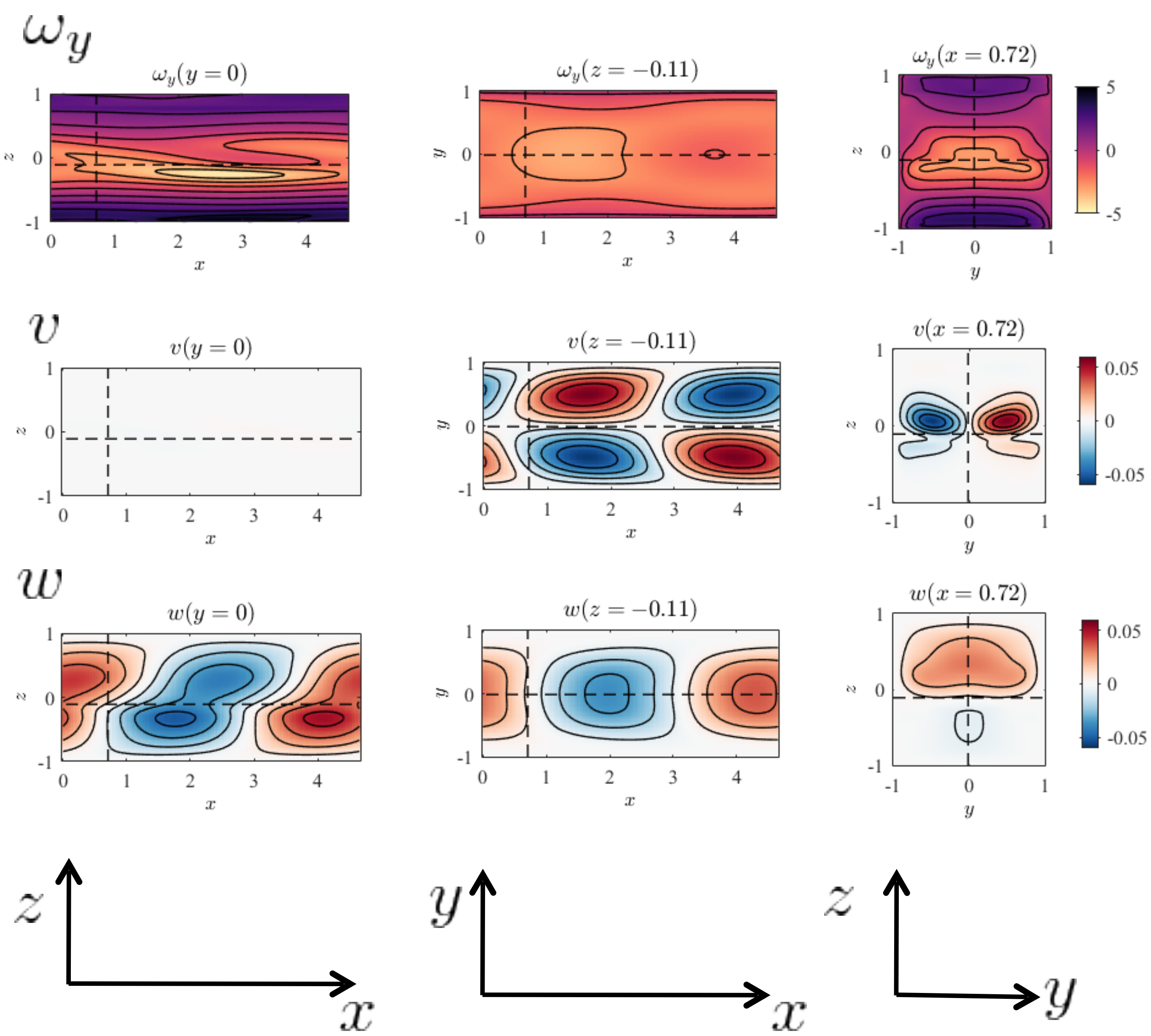

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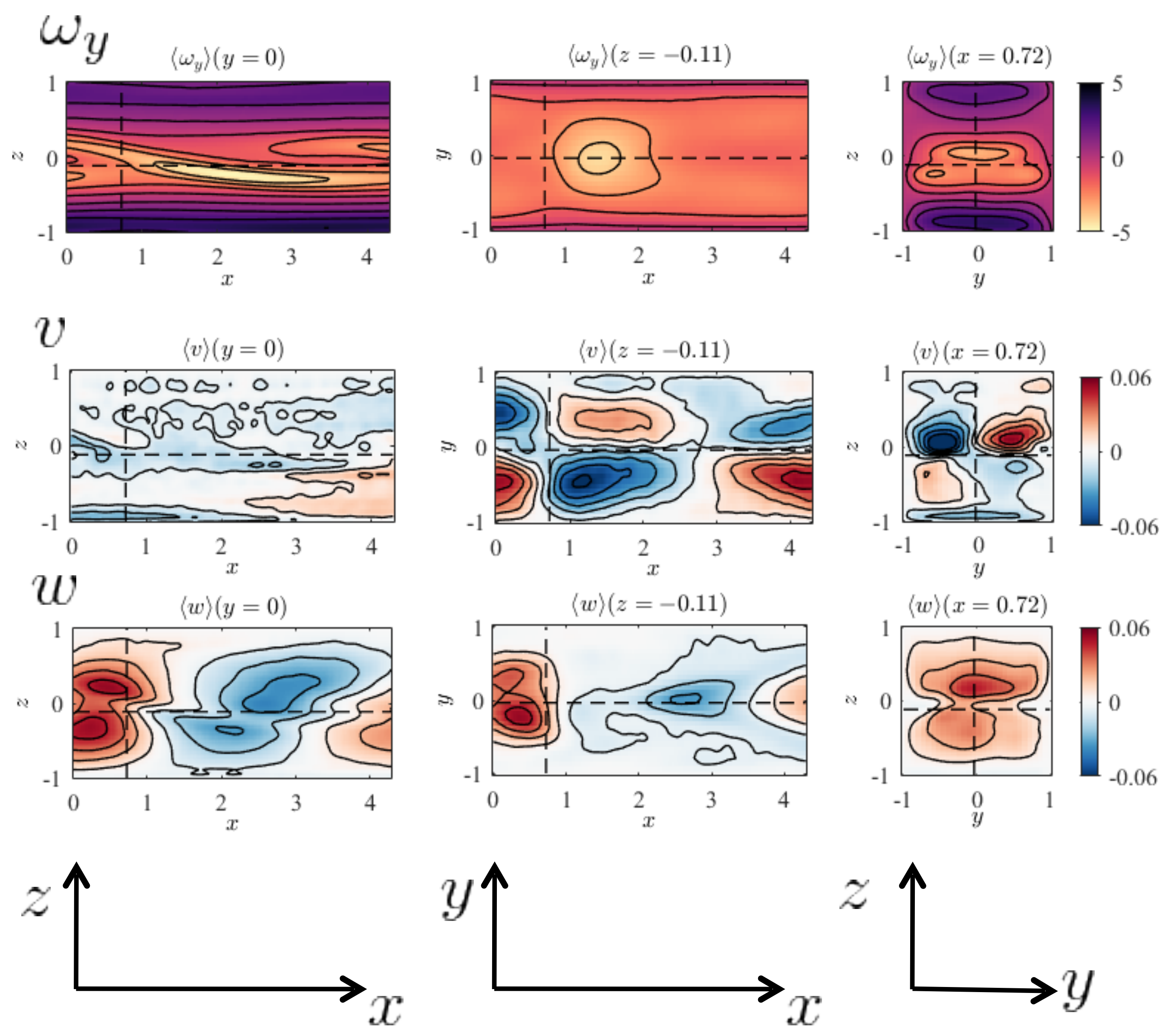
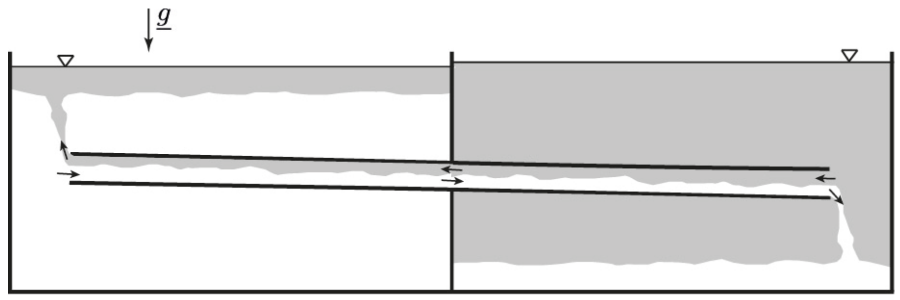
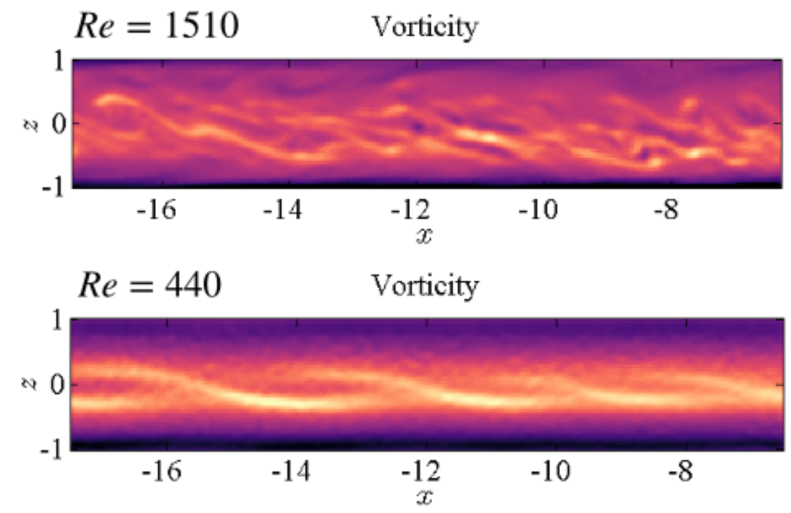


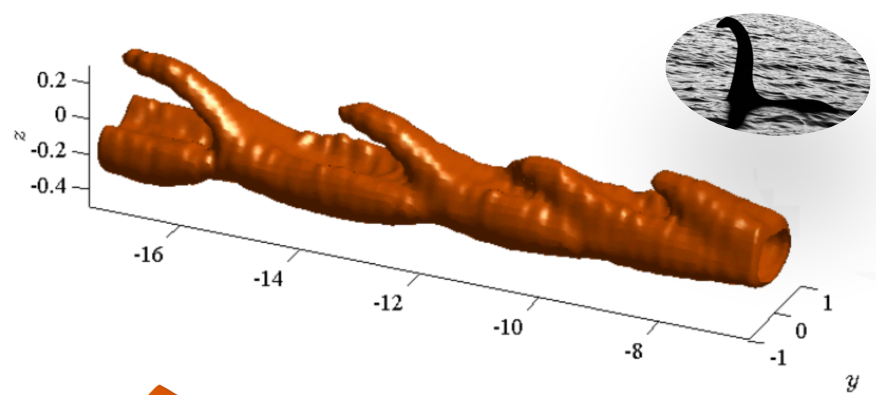
Summary

- The Stratified Inclined Duct experiment sets up a canonical **stratified shear flow**

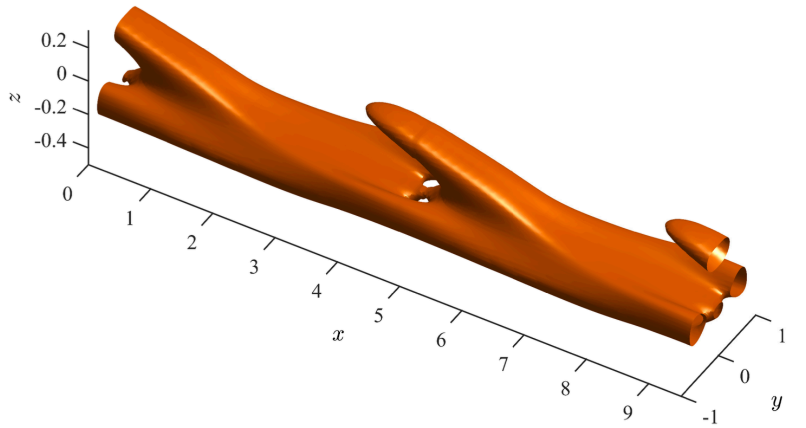




- **Volumetric measurements** reveal that these flows exhibit various **3D coherent structures** which are **increasingly complex** as Re is increased



- The low-Re structure results from the saturation of a **confined Holmboe instability**, predicted by a **3D stability analysis on the measured mean flow**



More details in: Lefauve, Partridge, Zhou, Caulfield, Dalziel & Linden *Journal of Fluid Mechanics* **848** : 508-544 (2018)